Long-Run Behavior and Convergence of Dynamic Mean Field Equilibria

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Abstract. We study the behavior of dynamic equilibria in mean field games with large time horizons in a dynamic consumer choice model. We show that if the stationary equilibrium in the associated infinite horizon game is unique, the dynamic equilibria of the finite horizon games converge to the stationary equilibrium of the infinite horizon game as the time horizon tends to infinity. If the stationary equilibrium is not unique, however, the situation becomes more involved. In this case, we show that in addition to convergence to the stationary equilibria it is possible that, in the long run, the dynamic equilibria circle around randomized stationary equilibria.

Keywords: mean field game, dynamic equilibrium, stationary equilibrium, turnpike property

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1. Introduction

The objective of this article is to study the convergence of equilibria of finite time horizon mean field games as the time horizon tends to infinity. To wit, one may expect that under reasonable assumptions the finite horizon equilibria converge to equilibria in the corresponding infinite horizon mean field game. In fact, there are several cases in which such convergence has been observed in the existing literature. What these results have in common is that assumptions are put in place which guarantee that the stationary equilibrium in the infinite horizon game is unique. With the present article, our aim is to shed light on the long-run behavior and convergence of finite horizon equilibria in cases in which uniqueness in the infinite horizon limit fails. The intuition in this setting is that there are multiple attractors for the finite horizon

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equilibria, and hence the limiting behavior is much more dependent on the boundary data imposed upon the finite horizon game. Adding to this, the absence of uniqueness gives rise to randomized stationary equilibria and it is a priori unclear what their effect on the long-run behavior of (non-randomized, finite horizon) equilibria may be.

To address these challenging questions, we consider a simple toy model motivated by consumer choice in which the representative agent problem has finite state and action spaces. The motivation for this choice is two-fold. First, games with finite state and action spaces are among the most tractable classes of mean field games. In particular, following e.g. [BHS21], equilibria in the finite horizon setting can be constructed by solving a system of forward-backward ordinary differential equations in contrast to systems of partial differential equations in the general case. Second and even more importantly, the restriction to a finite state space gives us access to the powerful results of [Neu20] which allow us to characterize all stationary equilibria (including randomized ones) of the infinite horizon game.

As it turns out, the stationary equilibria of the infinite horizon problem correspond to stationary points of the system of differential equations characterizing the finite horizon equilibria. In light of this, the task we face in this work is the study of convergence of a dynamical system to its stationary points. While problems of this type have a long history, the particular system which arises in our setting is non-standard and comes with its own unique challenges. To be precise, the system we consider takes the form

$$M(t) = F_{\rightarrow}\big(M(t), W(t)\big) \qquad \text{and} \qquad W(t) = F_{\leftarrow}\big(M(t), W(t)\big),$$

where $F_{\rightarrow}$ describes the forward dynamics (i.e. we impose an initial condition on M) and $F_{\leftarrow}$ describes the backward dynamics (meaning that we impose a terminal condition on W). The first challenge arises from the fact that this is a forward-backward system in contrast to purely forward systems usually studied in the dynamical systems literature. Second and more strikingly, the right-hand side of the forward equation, that is $F_{\rightarrow}$, turns out to be discontinuous. While there has been some progress on the study of dynamical systems with discontinuous right-hand sides in recent years, our particular system does not seem to be covered by the existing literature in that field.

The mean field game under consideration in this article has the property that the number of stationary points of the dynamical system depends crucially on the choice of parameters. That is, while one can choose the parameters so that the stationary point is unique (and hence so is the equilibrium in the infinite horizon problem), there are other parameter constellations which give rise to up to five stationary points corresponding to three non-randomized and two randomized stationary equilibria. We take advantage of this property and study two cases in great detail. In the first case, the parameters are chosen such that the stationary point is unique. Here, the problem ends up being quite tractable and hence it should not come as a surprise that we are able to establish convergence of the finite horizon equilibria to the unique stationary equilibrium of the infinite horizon game. Convergence is to be understood in the sense that there exists a time interval whose size grows to infinity in the limit, on which the dynamical system converges to the unique stationary point. By analogy, this property is referred to as the *turnpike property*; see [GZ22] for a recent overview and historical background. The second case we consider admits three stationary equilibria, one of which is randomized, and both the study of convergence and the results we obtain become significantly more sophisticated. Depending on the initial and terminal data of the dynamical system, we either observe convergence to one of the equilibria, that the finite horizon equilibrium circles around the randomized stationary equilibrium, or combinations of both. This signifies the tremendous effect the assumption of uniqueness has on the long-run behavior of equilibria and constitutes the main contribution of this article.

1.1. Related Literature

The beginnings of mean field game theory are usually accredited to the articles [HMC06] and [LL07]. Since then, both a rich mathematical theory has unfolded and many applications of that theory have emerged. We refer to the monographs [BFY13; CD18a; CD18b] or the lecture notes [Car13b] for the general theory and mention [BD15; GVW14; Gué09; KB16; KM17] as a non-exhaustive list of exemplary applications. The particular mean field game we consider in the present article falls into the class of continuous-time mean field games with finite state space, for which the general theory has been developed in [GMR13; Gué15; BC18; CP19b; DGG19; CF20; Neu20; BHS21; CW21] among others; see also §7.2 in [CD18a].

Convergence of equilibria in mean field games with finite state spaces has been established in [GMR10] in discrete time and [GMR13] in continuous time under the Lasry-Lions monotonicity condition (implying uniqueness of equilibria). Moreover, it has been established in the general diffusion-based setting under the monotonicity condition in [Car+12; Car+13] for quadratic Hamiltonians and in [Por18] for globally Lipschitz, locally uniformly convex Hamiltonians, respectively. Convergence results for weakly non-monotone games can be found in [CP21]. In addition to convergence of equilibria, there are results on convergence of the (scaled) value function to a limit (which does not necessarily coincide with an equilibrium value of the stationary game) under more general assumptions. We mention [CP19a] under the monotonicity condition, [Car13a] for first order games given a coercivity condition (which again implies uniqueness of the value of the stationary equilibrium), [FG14] for potential games with finite state spaces, [CM20; Mas19] for potential games in the diffusion setting, and [BK23] for games admitting a very particular attraction behavior. Finally, there is a series of papers [CC21; Cir19; CN18] in which periodic solutions for some non-monotone, first order mean field games are constructed. Let us also mention [KM18], in which a model linked to botnet defence and the spread of corruption is considered, and dynamic equilibria exhibiting the turnpike property are constructed from stationary equilibria. This is in contrast to our results, as we do not construct dynamic equilibria having the turnpike property, but in fact establish the long-run behavior of all dynamic equilibria.

Regarding the study of dynamical systems with discontinuous right-hand sides, we mention the classical monograph [Fil88] for a general overview. Moreover, for the study of stationary points of dynamical systems with discontinuous right-hand sides, we refer e.g. to the recent results obtained in [BCS13; DEP17]. However, let us highlight that we are not aware of any results in the literature which cover our particular system of equations, neither in terms of existence of solutions nor in terms of qualitative behavior of the dynamical system. As such, our work also contributes to this independent strand of literature.

The remainder of the paper is organized as follows. We first discuss the mean field game in Section 2 and introduce the forward-backward system of differential equations characterizing dynamic equilibria. In Section 3 we gather preliminary results on properties of the solution of the forward-backward system which allow us to derive the precise structure of the solution in the subsequent sections. Section 4 is then devoted to the study of the case with a unique stationary equilibrium, whereas Section 5 addresses the case with multiple stationary equilibria. Finally, Appendix A contains the preliminary analysis of the forward-backward system including an existence result and Appendix B is devoted to the proof that the forward-backward system indeed induces dynamic equilibria.

2. A Dynamic Consumer Choice Model

We consider a dynamic version of the stationary consumer choice model studied in [Neu20]. Employing the framework introduced in [BHS21], the model is formulated as a mean field game with finite time horizon $T > 0$. In this model, a continuum of agents has the choice between two phone providers and the individual agents' utility depends on the share of agents with the same provider.

To make this precise, we denote by $X_t \in \mathbb{S} := \{1, 2\}$ the provider of a representative agent at any time $t \in [0, T]$. The agent's action at time t is denoted by $\nu_t \in \mathbb{U} := \{stay, switch\}$, where we interpret $\nu_t = stay$ as the action of staying with the current provider and $\nu_t = switch$ as the action of switching providers. A switch between providers, i.e. a jump in $X = \{X_t\}_{t\in[0,T]}$, is not assumed to take place instantaneously but at transition rates instead. More precisely, if $\nu_t = switch$ the agent switches to the other provider at a rate of $\kappa > 0$, whereas if $\nu_t = stay$ a switch between providers only occurs at a small baseline transition rate $\varepsilon > 0$ satisfying $\varepsilon < \kappa$. Thus, writing

$$Q(u) := \begin{pmatrix} -\varepsilon & \varepsilon \\ \varepsilon & -\varepsilon \end{pmatrix} \mathbf{1}_{\{u=stay\}} + \begin{pmatrix} -\kappa & \kappa \\ \kappa & -\kappa \end{pmatrix} \mathbf{1}_{\{u=switch\}}, \qquad u \in \mathbb{U},$$

the time-t transition rate matrix of the state process X under the action ν_t is given by $Q(\nu_t)$. For simplicity, we restrict[1] the agent's actions to Markovian feedback controls given by

$$\mathcal{A}_T := \big\{\nu : [0, T] \times \mathbb{S} \to \mathbb{U} \;\big|\; \nu \text{ is Borel-measurable}\big\}.$$

To wit, given $\nu \in \mathcal{A}_T$, the agent's time-t action is

$$\nu_t := \nu(t, X_{t-}), \qquad t \in [0, T].$$

With this, we can construct a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ supporting an $\mathbb{S}$-valued process $X = \{X_t\}_{t\in[0,T]}$ such that for each $\nu \in \mathcal{A}_T$ there is a probability measure $\mathbb{P}^\nu \sim \mathbb{P}$ such that

$$\text{under } \mathbb{P}^\nu, \text{ the time-}t \text{ transition rate matrix of } X \text{ is } Q(\nu_t) = Q\big(\nu(t, X_{t-})\big);$$

[1]Following [BHS21], optimizers in the class of Markovian feedback controls are also optimal in the larger class of closed-loop controls and hence no generality is lost by this restriction.

see Section 2 in [BHS21] for details. Writing $\mathbb{E}^\nu$ for the expectation operator with respect to $\mathbb{P}^\nu$ and fixing a discount factor $\beta > 0$, the representative agent faces the optimization problem

$$\sup_{\nu \in \mathcal{A}_T} \mathbb{E}^\nu\Big[\int_0^T e^{-\beta t}\psi_{X_t}(M(t),\nu_t)\,\mathrm{d}t + e^{-\beta T}\Psi_{X_T}\big(M(T)\big)\Big]. \tag{2.1}$$

Here, the Borel-measurable function $M : [0,T] \to [0,1]$ models the fraction of agents serviced by provider $1 \in \mathbb{S}$ as a function of time, implying that $1 - M$ is the fraction of agents serviced by provider $2 \in \mathbb{S}$. The functions ψ and Ψ appearing in (2.1) model the running and terminal reward, respectively. The running reward $\psi : [0,1] \times \mathbb{U} \to \mathbb{R}^2$ is assumed to be of the form

$$\psi(m,u) = \begin{pmatrix} \psi_1(m,u) \\ \psi_2(m,u) \end{pmatrix} := \begin{pmatrix} U_1(m) - C\mathbf{1}_{\{u=switch\}} \\ U_2(1-m) - C\mathbf{1}_{\{u=switch\}} \end{pmatrix}, \qquad (m,u) \in [0,1] \times \mathbb{U},$$

where $C > 0$ denotes the instantaneous switching cost and $U_i(m)$ the utility of being serviced by provider $i \in \mathbb{S}$ given that the share of agents using the same provider is m. Following [Neu20], we assume U_i to be of the form

$$U_i : [0,1] \to \mathbb{R}, \qquad m \mapsto U_i(m) := \log\big(f_\delta(m)\big) + s_i$$

for $s_i \in \mathbb{R}$ and a cutoff function[2]

$$f_\delta : [0,1] \to [0,1], \qquad y \mapsto \begin{cases} y & \text{if } y \geq \delta \\ \frac{1}{2\delta}y^2 + \frac{\delta}{2} & \text{if } y < \delta \end{cases}$$

for some $\delta \in (0,1)$ small. Finally, we suppose that the terminal reward

$$\Psi : [0,1] \to \mathbb{R}^2, \qquad m \mapsto \Psi(m) = \begin{pmatrix} \Psi_1(m) \\ \Psi_2(1-m) \end{pmatrix}$$

is continuous and hence bounded. With this, we speak of a *mean field equilibrium* if we can find a pair (ν, M) such that $\nu \in \mathcal{A}_T$ is optimal for (2.1) and the distribution of X under the optimal action ν coincides with M, that is

$$\mathbb{P}^\nu\big[X_t \in \cdot\,\big] = \big(M(t), 1 - M(t)\big), \qquad t \in [0,T].$$

In other words, equilibrium obtains if the agent's ex ante expectations coincide with the ex post aggregrate distribution resulting from all agents' optimal decisions.

2.1. Construction of Dynamic Equilibria

Since the model is formulated in the framework of [BHS21], the construction of equilibria can be reduced to solving a system of forward-backward ordinary differential equations. The forward equation describes the evolution of the aggregate distribution of agents $M(t)$ in equilibrium,

[2]The cutoff function is just a technical tool needed to ensure $U_i(0) > -\infty$.

whereas the backward equation determines the value function $v : [0,T] \times \mathbb{S} \to \mathbb{R}$ associated with the representative agent problem (2.1) in equilibrium.

Before introducing the forward-backward system we observe that, to wit, the optimal action of the representative agent depends only on the difference between the indirect utilities of being with the respective providers. More precisely, if $v(t,i)$ denotes the value function at time $t \in [0,T]$ conditioned on the representative agent being in state $i \in \mathbb{S}$, we argue below that optimal actions only depend on the difference

$$W(t) := \big[v(t,1) - v(t,2)\big]e^{\beta t}, \qquad t \in [0,T].$$

The optimal action in state $i = 1$ is to switch providers if and only if $W(t)$ is below a certain threshold, whereas the optimal action in state $i = 2$ is to switch providers if and only if $W(t)$ is above another threshold. We will see below that these two thresholds are given by $-\rho$ and ρ, respectively, where

$$\rho := \tfrac{C}{\kappa-\varepsilon}.$$

With these ideas fixed, using the results of [BHS21], we can now introduce the system of forward-backward ordinary differential equations whose solutions describe dynamic mean field equilibria as detailed in Theorem 2.1 below. We look for functions

$$M : [0,T] \to [0,1] \qquad \text{and} \qquad W : [0,T] \to \mathbb{R}$$

which solve the system of differential equations

$$\dot{M}(t) = F_{\rightarrow}\big(M(t), W(t)\big), \qquad t \in [0,T], \qquad M(0) = M_0, \tag{2.2}$$

$$\dot{W}(t) = F_{\leftarrow}\big(M(t), W(t)\big), \qquad t \in [0,T], \qquad W(T) = G\big(M(T)\big). \tag{2.3}$$

We refer to these equations as the *equilibrium system*. The functions $F_{\rightarrow} : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $F_{\leftarrow} : [0,1] \times \mathbb{R} \to \mathbb{R}$ are given by

$$F_{\rightarrow}(m,w) := \begin{cases} \varepsilon - (\kappa+\varepsilon)m & \text{if } w \le -\rho, \\ \varepsilon - 2\varepsilon m & \text{if } w \in (-\rho,\rho), \\ \kappa - (\kappa+\varepsilon)m & \text{if } w \ge \rho, \end{cases} \qquad (m,w) \in [0,1] \times \mathbb{R}, \tag{2.4}$$

$$F_{\leftarrow}(m,w) := \begin{cases} g(m) + C + (\beta+\kappa+\varepsilon)w & \text{if } w \le -\rho, \\ g(m) + (\beta+2\varepsilon)w & \text{if } w \in (-\rho,\rho), \\ g(m) - C + (\beta+\kappa+\varepsilon)w & \text{if } w \ge \rho, \end{cases} \qquad (m,w) \in [0,1] \times \mathbb{R}. \tag{2.5}$$

Moreover, in (2.2), $M_0 \in [0,1]$ denotes the initial fraction of agents in state $i = 1$ at time $t = 0$ whereas the terminal condition in (2.3) is given as the difference in terminal rewards in the respective states, that is

$$G : [0,1] \to \mathbb{R}, \qquad m \mapsto G(m) := \Psi_1(m) - \Psi_2(1-m).$$

Finally, the function g appearing in $F_{\leftarrow}$ is given by the difference in running utility, that is

$$g : [0,1] \to \mathbb{R}, \qquad m \mapsto g(m) := U_2(1-m) - U_1(m) = \log\Big(\tfrac{f_\delta(1-m)}{f_\delta(m)}\Big) - (s_1 - s_2).$$

The case distinctions in (2.4) and (2.5) arise from the different optimal actions depending on whether $W(t)$ is below, in between, or above the two thresholds $-\rho$ and ρ. Let us furthermore highlight that $F_{\rightarrow}$ in (2.4) is discontinuous across $w = -\rho$ and $w = \rho$. As a consequence, we cannot expect (2.2) – (2.3) to admit solutions in the classical sense, but need to rely on a weaker solution concept instead. It turns out that we should interpret the forward equation in the Filippov sense [Fil88], which is to say that M is absolutely continuous and

$$\dot{M}(t) \in \begin{cases} \big\{F_{\rightarrow}\big(M(t), W(t)\big)\big\} & \text{if } W(t) \neq -\rho, \rho \\ \big[\varepsilon - (\kappa+\varepsilon)M(t), \varepsilon - 2\varepsilon M(t)\big] & \text{if } W(t) = -\rho \\ \big[\varepsilon - 2\varepsilon M(t), \kappa - (\kappa+\varepsilon)M(t)\big] & \text{if } W(t) = \rho \end{cases} \qquad \text{for a.e. } t \in [0,T].$$

The backward equation, on the other hand, can be interpreted in the classical sense, that is, W is continuously differentiable and satisfies

$$\dot{W}(t) = F_{\leftarrow}\big(M(t), W(t)\big) \qquad \text{for all } t \in [0,T].$$

We refer to Appendix A for a detailed account on existence, basic structural properties, and (local) closed-form expressions of M and W. Given (M, W), we can formalize the construction of (non-randomized) dynamic equilibria in the following theorem; the proof is given in Appendix B.

Theorem 2.1 (Dynamic Equilibrium). *Suppose that there exists a solution (M, W) of the equilibrium system* (2.2) – (2.3) *such that*

$$\big\{t \in [0,T] : W(t) \in \{-\rho, \rho\}\big\} \text{ has Lebesgue measure zero}$$

and define

$$h : \mathbb{R} \to \mathbb{U}^2, \qquad w \mapsto h(w) := \begin{cases} (\mathit{switch}, \mathit{stay}) & \text{if } w \leq -\rho \\ (\mathit{stay}, \mathit{stay}) & \text{if } w \in (-\rho, \rho) \\ (\mathit{stay}, \mathit{switch}) & \text{if } w \geq \rho. \end{cases}$$

Writing $h = (h_1, h_2)$, a mean field equilibrium for the dynamic consumer choice problem is given by the pair (ν, M) with

$$\nu : [0,T] \times \mathbb{S} \to \mathbb{U}, \qquad (t,i) \mapsto \nu(t,i) := h_i\big(W(t)\big). \qquad \diamond$$

The question of whether or not the set $\{W \in \{-\rho, \rho\}\}$ has Lebesgue measure zero depends on the model parameters and is intimately related to the existence of randomized equilibria. Indeed, since g is strictly decreasing, it has a strictly decreasing inverse denoted by g^{-1}. With this, we can introduce two constants

$$k_1 := g^{-1}\big((\beta + 2\varepsilon)\rho\big) \qquad \text{and} \qquad k_2 := g^{-1}\big(-(\beta+2\varepsilon)\rho\big). \tag{2.6}$$

Note that $k_1 < k_2$ by monotonicity of g^{-1}. In Lemma A.3 in the appendix we show that

$$\big\{t \in [0,T] : W(t) = -\rho\big\} \qquad \text{has Lebesgue measure zero if} \qquad k_1 \notin \big[\tfrac{\varepsilon}{\kappa+\varepsilon}, \tfrac{1}{2}\big],$$

$$\{t \in [0,T] : W(t) = \rho\} \qquad \text{has Lebesgue measure zero if} \qquad k_2 \notin \left[\tfrac{1}{2}, \tfrac{\kappa}{\kappa+\varepsilon}\right].$$

Now suppose that $k_1 \in [\varepsilon/(\kappa+\varepsilon), 1/2]$. Then it is possible that $\{W = -\rho\}$ has positive Lebesgue measure. The reason is that in this situation the point $(k_1, -\rho)$ is a stationary point of the equilibrium system (2.2) – (2.3). In fact, if $(M(t), W(t)) = (k_1, -\rho)$, the representative agent needs to randomize in order to achieve an equilibrium. More precisely, an equilibrium strategy requires the choice of actions

$$(\mathit{switch}, \mathit{stay}) \text{ with probability } p_- \quad \text{and} \quad (\mathit{stay}, \mathit{stay}) \text{ with probability } 1 - p_-,$$

where p_- solves

$$\begin{aligned} 0 &= p_- \lim_{w \uparrow -\rho} F_\rightarrow(k_1, w) + (1-p) \lim_{w \downarrow -\rho} F_\rightarrow(k_1, w) \\ &= p_- \big[\varepsilon - (\kappa+\varepsilon)k_1\big] + (1-p_-)\big[\varepsilon - 2\varepsilon k_1\big]. \end{aligned}$$

Note that this equation has a $[0,1]$-valued solution if and only if $k_1 \in [\varepsilon/(\kappa+\varepsilon), 1/2]$. Similarly, if $k_2 \in [1/2, \kappa/(\kappa+\varepsilon)]$, the set $\{W = \rho\}$ may have positive Lebesgue measure since (k_2, ρ) is a stationary point of the equilibrium system. The randomized equilibrium action in this case is

$$(\mathit{stay}, \mathit{stay}) \text{ with probability } p_+ \quad \text{and} \quad (\mathit{stay}, \mathit{switch}) \text{ with probability } 1 - p_+,$$

where p_+ solves

$$0 = p_+\big[\varepsilon - 2\varepsilon k_2\big] + (1-p_+)\big[\kappa - (\kappa+\varepsilon)k_2\big]. \tag{2.7}$$

As above, this equation has a $[0,1]$-valued solution if and only if $k_2 \in [1/2, \kappa/(\kappa+\varepsilon)]$. These arguments and the construction of randomized equilibria can be made rigorous but require a more general model setup compared to [BHS21] in which one allows randomized controls as well. Since this is outside the main focus of this article, we refrain from presenting this construction here.

2.2. The Stationary Model and its Solution

The objective of this article is to study the behavior of the dynamic equilibrium constructed in Theorem 2.1 as $T \to \infty$. Informally, the representative agent's optimization problem in the limit reads

$$\sup_{\nu \in \mathcal{A}_\infty} \mathbb{E}^\nu\Big[\int_0^\infty e^{-\beta t} \psi_{X_t}\big(M(t), \nu_t\big)\, \mathrm{d}t\Big].$$

Allowing for randomized strategies, that is, allowing the agent to choose their actions from the set $\mathcal{P}(\mathbb{U})$ of probability distributions over $\mathbb{U}$, the model fits into the framework of [Neu20]. As is customary in infinite horizon problems, we only look for stationary equilibria, i.e. we look for (randomized strategy) equilibria

$$M \in [0,1] \qquad \text{and} \qquad \nu : \mathbb{S} \to \mathcal{P}(\mathbb{U})$$

which do not depend on the time component. From this, we see that non-randomized stationary equilibria (M, ν) can be expressed as pairs $(M, h) \in [0,1] \times \mathbb{U}^2$ if we identify $h = (\nu(1), \nu(2))$.

In [Neu20], all stationary equilibria of the infinite horizon problem have been characterized, including all randomized strategy equilibria. The number and locations of the stationary equilibria are determined by the two constants k_1 and k_2 given in Equation (2.6) relative to the constants $\varepsilon/(\kappa+\varepsilon)$, $1/2$, and $\kappa/(\kappa+\varepsilon)$. To simplify the exposition and to avoid having to deal with too many case distinctions, we subsequently restrict our considerations to the following two representative cases.

(E1) $k_1 < \varepsilon/(\kappa+\varepsilon)$ and $k_2 < 1/2$.
In this situation there is a unique stationary equilibrium at $M = \kappa/(\kappa+\varepsilon)$ with associated equilibrium action $h = (stay, switch)$.

(E2) $k_1 < \varepsilon/(\kappa+\varepsilon)$ and $1/2 < k_2 < \kappa/(\kappa+\varepsilon)$.
In this case there are two non-randomized stationary equilibria at $M = 1/2$ with associated equilibrium action $h = (stay, stay)$ and at $M = \kappa/(\kappa+\varepsilon)$ with associated action $h = (stay, switch)$, respectively. There furthermore exists one randomized equilibrium, in which the representative agent chooses the action $(stay, stay)$ with probability $p \in (0, 1)$ and the action $(stay, switch)$ with probability $1 - p$. The equilibrium distribution of agents associated with this randomized equilibrium is $M = k_2$. The probability p is known explicitly and given as the unique probability such that $(k_2, 1 - k_2)$ is a stationary point given the equilibrium strategy, which means that p is the unique solution of

$$0 = k_2 \cdot (-\varepsilon) + (1 - k_2)\big(\varepsilon p + \kappa(1 - p)\big),$$

which in turn is equivalent to (2.7).

For both cases, we study the question of whether the dynamic equilibrium of the finite horizon problem obtained in Theorem 2.1 converges to an equilibrium of the infinite horizon problem as $T \to \infty$. For this, we need to study the behavior of solutions (M, W) of the equilibrium system (2.2) – (2.3) as $T \to \infty$. Note that while we focus only on the two specific parameter configurations (E1) and (E2), our arguments carry over to all other cases as well.

3. Properties of Solutions of the Equilibrium System

Before we study the asymptotic behavior of solutions (M, W) of the equilibrium system, we first analyze the properties of such solutions for a fixed finite time horizon T. In particular, it is crucial to understand how much time the solution (M, W) spends in the respective cases in the equilibrium system (2.2) – (2.3) and what happens at the transitions between them.

The main existence result on solutions of the equilibrium system is given in Theorem A.4 in the appendix. There, we establish existence of a solution (M, W) of the equilibrium system and show that there exists an index set $\mathcal{N} \subseteq \mathbb{N}$ and disjoint open intervals $\{I_n\}_{n\in\mathcal{N}}$ with $\bigcup_{n\in\mathcal{N}} I_n$ dense in $[0, T]$ such that, for each $n \in \mathcal{N}$, M is continuously differentiable on I_n and one of the following five cases holds:

$$\begin{aligned}
&\text{(Lo)} && W < -\rho && \text{and} && \dot{M} = \varepsilon - (\kappa+\varepsilon)M && \text{on } I_n,\\
&\text{(Med)} && W \in (-\rho, \rho) && \text{and} && \dot{M} = \varepsilon - 2\varepsilon M && \text{on } I_n,
\end{aligned}$$

$$
\begin{array}{llcll}
\text{(Hi)} & W > \rho & \text{and} & \dot{M} = \kappa - (\kappa + \varepsilon)M & \text{on } I_n, \\
(\text{Eq}^-) & W = -\rho & \text{and} & M = k_1 & \text{on } I_n, \\
(\text{Eq}^+) & W = \rho & \text{and} & M = k_2 & \text{on } I_n.
\end{array}
$$

In what follows, we shall always assume that the intervals I_n are maximal with respect to the properties in the respective cases.[3] With this, it follows that

$$
W(t) \in \{-\rho, \rho\} \qquad \text{for all } t \in (0,T) \setminus \bigcup_{n \in \mathcal{N}} I_n,
$$

i.e. W necessarily takes the value $-\rho$ or ρ at the transitions between the intervals I_n and there are at most countably many transitions. Finally, we note that on each interval I_n, both M and W satisfy linear differential equations and hence admit closed-form expressions which we have gathered in Appendix A for the reader's convenience and which we will subsequently refer to quite frequently.

When studying the two parameter constellations (E1) and (E2), the first objective is always to understand the precise sequence of intervals (Lo), (Med), (Hi), (Eq^-), and (Eq^+) the solution (M, W) of the equilibrium system traverses. For example, in case of (E1) we are able to show that there are at most the five different scenarios given by

$$
\begin{array}{c}
(\text{Med}) \qquad (\text{Lo}) \to (\text{Med}) \qquad (\text{Hi}) \to (\text{Med}) \\
(\text{Med}) \to (\text{Hi}) \to (\text{Med}) \qquad (\text{Lo}) \to (\text{Med}) \to (\text{Hi}) \to (\text{Med})
\end{array}
$$

The remainder of this section is devoted to establishing several preliminary results which facilitate the derivation of these scenarios. For this, we subsequently fix $n \in \mathcal{N}$ and a corresponding interval I_n. We furthermore suppose that $I_n = (a, b)$ for $a, b \in [0, T]$ with $a < b$. We proceed to analyze the properties of (M, W) on (a, b), beginning with a relatively simple result which gives information on the value of M at the beginning of the interval (a, b).

Lemma 3.1. *Suppose that $a > 0$. Then the following statements hold.*

1. *If (a, b) is of type (Lo), then $M(a) \geq k_1$.*
2. *If (a, b) is of type (Med) and $W(a) = -\rho$, then $M(a) \leq k_1$.*
3. *If (a, b) is of type (Med) and $W(a) = \rho$, then $M(a) \geq k_2$.*
4. *If (a, b) is of type (Hi), then $M(a) \leq k_2$.* $\diamond$

Proof. The idea is the same for all four cases, so we only prove the first one. That is, we assume that (a, b) is of type (Lo), i.e. $W(t) < -\rho$ for all $t \in (a, b)$. We note that in this situation we must have $W(a) = -\rho$ by continuity of W, maximality of (a, b), and since $a > 0$. The two properties $W(a) = -\rho$ and $W < -\rho$ on (a, b) show that $\dot{W}(a) \leq 0$. Moreover, at a we are in the first case of (2.5) and hence

$$
0 \geq \dot{W}(a) = g\big(M(a)\big) + C + (\beta + \kappa + \varepsilon)W(a) = g\big(M(a)\big) - (\beta + 2\varepsilon)\rho,
$$

[3] For example, if I_n is of type (Lo), maximality is to be understood in the sense that there is no open interval $J \subset [0, T]$ such that $W < \rho$ on J and I_n is a proper subset of J.

where we have used $W(a) = -\rho$ and $C = (\kappa - \varepsilon)\rho$ for the last identity. Solving for $M(a)$ yields

$$M(a) \geq g^{-1}\big((\beta + 2\varepsilon)\rho\big) = k_1,$$

which concludes the proof. $\square$

The next result provides conditions under which we can conclude that $a = 0$.

Lemma 3.2. *The following statements hold.*

1. *If (a, b) is of type* (Lo) *and $M(t) < k_1$ for all $t \in (a, b)$, then $a = 0$.*
2. *If (a, b) is of type* (Med) *and $M(t) \in (k_1, k_2)$ for all $t \in (a, b)$, then $a = 0$.*
3. *If (a, b) is of type* (Hi) *and $M(t) > k_2$ for all $t \in (a, b)$, then $a = 0$.* $\diamond$

Proof. As before, we only give a proof for the case in which (a, b) is of type (Lo) and $M(t) < k_1$ for all $t \in (a, b)$ since the other two cases follow by similar arguments. As (a, b) is of type (Lo), it follows that W is given explicitly as

$$W(a) = W(b)e^{-(\beta+\kappa+\varepsilon)(b-a)} - \int_a^b e^{(\beta+\kappa+\varepsilon)(a-t)}\big[g\big(M(t)\big) + C\big]\,\mathrm{d}t;$$

see Equation (A.7) in the appendix. Since $W(b) \leq -\rho$ by continuity of W (with equality if $b < T$), it follows that

$$W(a) \leq -\rho e^{-(\beta+\kappa+\varepsilon)(b-a)} - \int_a^b e^{(\beta+\kappa+\varepsilon)(a-t)}\big[g\big(M(t)\big) + C\big]\,\mathrm{d}t.$$

Since $M < k_1$ on (a, b) and the function g is strictly decreasing, we find that

$$g\big(M(t)\big) + C > g(k_1) + C = (\beta + 2\varepsilon)\rho + C = (\beta + \kappa + \varepsilon)\rho, \qquad t \in (a, b),$$

where we have used that $C = (\kappa - \varepsilon)\rho$ and that $k_1 = g^{-1}((\beta + 2\varepsilon)\rho)$; see Equation (2.6). Thus, since $a < b$,

$$W(a) < -\rho e^{-(\beta+\kappa+\varepsilon)(b-a)} - \int_a^b e^{(\beta+\kappa+\varepsilon)(a-t)}(\beta + \kappa + \varepsilon)\rho\,\mathrm{d}t = -\rho.$$

Now if a were strictly positive, continuity of W and maximality of (a, b) for $W < -\rho$ would imply that $W(a) = -\rho$. In light of the strict inequality above, we conclude that $a = 0$. $\square$

Whenever $W(t)$ is either equal to $-\rho$ and ρ, it will be important to decide in which case we were before time t just by looking at $M(t)$. The next result provides sufficient conditions under which this is possible.

Lemma 3.3. *The following statements hold.*

1. *If $W(b) = -\rho$ and $M(b) < k_1$, the interval (a, b) is of type* (Lo).
2. *If $W(b) = -\rho$ and $M(b) > k_1$, the interval (a, b) is of type* (Med).

3. *If* $W(b) = \rho$ *and* $M(b) < k_2$*, the interval* (a, b) *is of type* (Med).
4. *If* $W(b) = \rho$ *and* $M(b) > k_2$*, the interval* (a, b) *is of type* (Hi). $\diamond$

Proof. As usual, we only consider the first case as the other cases can be proved analogously. Since W is continuously differentiable, we note that it satisfies

$$\dot{W}(b) = g\big(M(b)\big) + C + (\beta + \kappa + \varepsilon)W(b) = g\big(M(b)\big) - (\beta + 2\varepsilon)\rho,$$

since $W(b) = -\rho$ and $C = (\kappa - \varepsilon)\rho$. As g is strictly decreasing and $M(b) < k_1$, it follows that

$$\dot{W}(b) > g(k_1) - (\beta + 2\varepsilon)\rho = 0$$

by definition of k_1; see (2.6). This concludes the proof by continuity of W and $\dot{W}$. $\square$

4. The Unique Stationary Equilibrium Case (E1)

We now consider the case (E1) in detail. That is, we subsequently assume that $k_1 < \varepsilon/(\kappa + \varepsilon)$ and $k_2 < 1/2$. Recall that this implies that there exists a unique stationary equilibrium at $M = \kappa/(\kappa + \varepsilon)$ with associated equilibrium strategy $h = (stay, switch)$. Moreover, according to Theorem A.4 in the appendix, the assumptions on k_1 and k_2 imply that there are no intervals of type (Eq^-) or (Eq^+), so that $\{W \in \{-\rho, \rho\}\}$ is a countable subset of $[0, T]$. Finally, to keep the number of case distinctions at a reasonable level, we furthermore assume for convenience that the terminal condition satisfies

$$-\rho < G(m) < \rho, \qquad m \in [0, 1]. \tag{4.1}$$

This assumption guarantees that the last interval before T is of type (Med).

4.1. Structure of the Solution (M, W)

We now give a complete characterization of the structure of the solution (M, W) of the equilibrium system in the sense that we derive all possible sequences of cases that the solution traverses. For this, a backward induction can be used to characterize the structure of the solution (M, W). More precisely, we observe that (4.1) implies that there exists $t_1 \in [0, T)$ such that the interval (t_1, T) is of type (Med). At this point, we need to distinguish several cases as follows.

- ▷ $t_1 = 0$, in which case $(0, T)$ is of type (Med).
- ▷ $t_1 > 0$ and $W(t_1) = -\rho$. In this case, there exists some $t_2 \in [0, t_1)$ such that the interval (t_2, t_1) is of type (Lo). We claim that we must in fact have $t_2 = 0$. Indeed, let us argue by contradiction and assume that $t_2 > 0$. In this situation, Lemma 3.1.1 and 3.1.2 are applicable, so we conclude that $M(t_1) \leq k_1 \leq M(t_2)$. However, (t_2, t_1) is of type (Lo) and hence M is given explicitly as

  $$M(t) = \Big(M(t_1) - \tfrac{\varepsilon}{\kappa+\varepsilon}\Big)e^{(\kappa+\varepsilon)(t_1 - t)} + \tfrac{\varepsilon}{\kappa+\varepsilon}, \qquad t \in [t_2, t_1];$$

  see Equation (A.6) in the appendix. Since $M(t_1) \leq k_1 < \varepsilon/(\kappa + \varepsilon)$, it follows that M is strictly increasing on $[t_2, t_1]$, which contradicts $M(t_1) \leq k_1 \leq M(t_2)$. We have therefore argued that $t_2 = 0$.

▷ $t_1 > 0$ and $W(t_1) = \rho$. It follows that there exists $t_2 \in [0, t_1)$ such that the interval (t_2, t_1) is of type (Hi) and we have to consider two subcases.

▷▷ $t_2 = 0$, in which case the backward induction ends.

▷▷ $t_2 > 0$, in which case there exists $t_3 \in [0, t_2)$ such that the interval (t_3, t_2) is of type (Med). Once again, we have to consider two subcases.

▷▷▷ $t_3 = 0$, in which case the backward induction ends.

▷▷▷ $t_3 > 0$, in which case $(0, t_3)$ is of type (Lo) and the backward induction ends as well. To see this, we first rule out the possibility of ending up with an interval of type (Hi). Since we know that (t_3, t_2) is of type (Med), it follows that

$$M(t) = \Big(M(t_2) - \tfrac{1}{2}\Big)e^{2\varepsilon(t_2 - t)} + \tfrac{1}{2}, \qquad t \in [t_3, t_2];$$

see Equation (A.9). Next, since (t_2, t_1) is of type (Hi), Lemma 3.1.4 shows that $M(t_2) \leq k_2 < 1/2$. Thus M is strictly increasing on $[t_3, t_2]$, that is $M(t_3) < M(t_2) \leq k_2$. Now assume by contradiction that $W(t_3) = \rho$. Then we may apply Lemma 3.1.3 to the type (Med) interval (t_3, t_2), which immediately yields the contradiction $M(t_3) \geq k_2$. We have therefore argued that there exists $t_4 \in [0, t_3)$ such that (t_4, t_3) is an interval of type (Lo) and we are left with showing that $t_4 = 0$. To see this, note that M is given explicitly as

$$M(t) = \Big(M(t_3) - \tfrac{\varepsilon}{\kappa+\varepsilon}\Big)e^{(\kappa+\varepsilon)(t_3 - t)} + \tfrac{\varepsilon}{\kappa+\varepsilon}, \qquad t \in [t_4, t_3];$$

see Equation (A.6). Since (t_3, t_2) is of type (Med) and $W(t_3) = -\rho$, Lemma 3.1.2 shows that $M(t_3) \leq k_1 < \varepsilon/(\kappa + \varepsilon)$, so M is strictly increasing on $[t_4, t_3]$. But then it follows that $M(t) < k_1$ on (t_4, t_3) and thus $t_4 = 0$ by Lemma 3.2.1.

Summing up the above discussion, we have argued that only the following five sequences of interval types are possible:

$$\text{(Med)} \qquad \text{(Lo)} \to \text{(Med)} \qquad \text{(Hi)} \to \text{(Med)}$$
$$\text{(Med)} \to \text{(Hi)} \to \text{(Med)} \qquad \text{(Lo)} \to \text{(Med)} \to \text{(Hi)} \to \text{(Med)}$$

4.2. Convergence of Dynamic Equilibria

Having identified all possible scenarios, we can now analyze the behavior of the solution (M, W) as $T \to \infty$. The main tool in our analysis is the following result.

Proposition 4.1. *Let (a, b) be a non-empty open subinterval of $[0, T]$.*

1. *Suppose that $k_1 < \varepsilon/(\kappa + \varepsilon)$. If (a, b) is of type (Lo) and $W(b) = -\rho$, there exists a constant $L > 0$ which does not depend on T such that $b - a \leq L$.*
2. *Suppose that $k_2 < 1/2$. If (a, b) is of type (Med), there exists a constant $L > 0$ which does not depend on T such that $b - a \leq L$.* ◇

Proof. 1. Suppose that $k_1 < \varepsilon/(\kappa+\varepsilon)$, the interval (a,b) is of type (Lo), and $W(b) = -\rho$. In this case, M is given explicitly by

$$M(t) = \Big(M(a) - \tfrac{\varepsilon}{\kappa+\varepsilon}\Big)e^{-(\kappa+\varepsilon)(t-a)} + \tfrac{\varepsilon}{\kappa+\varepsilon}, \qquad t \in [a,b];$$

see Equation (A.5). If we define

$$\hat{M} : [0,\infty) \to \mathbb{R}, \qquad t \mapsto \hat{M}(t) := \tfrac{\varepsilon}{\kappa+\varepsilon}\Big(1 - e^{-(\kappa+\varepsilon)(t-a)}\Big),$$

it follows that

$$M(t) - \hat{M}(t) = M(a)e^{-(\kappa+\varepsilon)(t-a)} \geq 0, \qquad t \in [a,b].$$

The function $\hat{M}$ is strictly increasing and satisfies

$$\lim_{t\to\infty} \hat{M}(t) = \tfrac{\varepsilon}{\kappa+\varepsilon} > k_1.$$

As such, there exists $L > 0$ which does not depend on a, b, T such that $\hat{M}(L+a) > k_1$. We subsequently argue by contradiction and assume that $b - a > L$. Then monotonicity of $\hat{M}$ implies that

$$M(b) \geq \hat{M}(b) = \hat{M}(b-a+a) > \hat{M}(L+a) > k_1$$

and hence, by continuity of M, we find $t_0 \in (a,b)$ such that $M(t_0) > k_1$. Since g is strictly decreasing and by definition of k_1 in (2.6) it follows that

$$g\big(M(t_0)\big) < g(k_1) = (\beta + 2\varepsilon)\rho,$$

from which we conclude using $C = (\kappa - \varepsilon)\rho$ that

$$\eta := \rho - \frac{g\big(M(t_0)\big) + C}{\beta+\kappa+\varepsilon} > \rho - \frac{(\beta+2\varepsilon)\rho + C}{\beta+\kappa+\varepsilon} = 0.$$

Next, since $W(b) = -\rho$ and (a,b) is of type (Lo), it follows from Lemma 3.3.2 that we must have $M(b) \leq k_1 < \varepsilon/(\kappa+\varepsilon)$. Now according to Equation (A.6) we have

$$M(t) = \Big(M(b) - \tfrac{\varepsilon}{\kappa+\varepsilon}\Big)e^{(\kappa+\varepsilon)(b-t)} + \tfrac{\varepsilon}{\kappa+\varepsilon}, \qquad t \in [a,b],$$

from which we therefore conclude that M is strictly increasing on (a,b). Moreover, as g is strictly decreasing and M is strictly increasing, it follows that $g(M)$ is strictly decreasing on (a,b). Using Equation (A.7) for the explicit representation of W on (a,b), we conclude that

$$\begin{aligned} W(t_0) &= W(b)e^{-(\beta+\kappa+\varepsilon)(b-t_0)} - \int_{t_0}^{b} e^{(\beta+\kappa+\varepsilon)(t_0-s)}\big[g\big(M(s)\big) + C\big]\,\mathrm{d}s \\ &> W(b)e^{-(\beta+\kappa+\varepsilon)(b-t_0)} - \big[g\big(M(t_0)\big) + C\big]\int_{t_0}^{b} e^{(\beta+\kappa+\varepsilon)(t_0-s)}\,\mathrm{d}s. \end{aligned}$$

Computing the integral, using that $W(b) = -\rho$ by assumption, and employing the definition of $\eta > 0$, it follows that

$$
\begin{aligned}
W(t_0) &> -\rho e^{-(\beta+\kappa+\varepsilon)(b-t_0)} - \frac{g\big(M(t_0)\big) + C}{\beta+\kappa+\varepsilon}\Big(1 - e^{-(\beta+\kappa+\varepsilon)(b-t_0)}\Big) \\
&= -\rho e^{-(\beta+\kappa+\varepsilon)(b-t_0)} + (\eta - \rho)\Big(1 - e^{-(\beta+\kappa+\varepsilon)(b-t_0)}\Big) \\
&= -\rho + \eta\Big(1 - e^{-(\beta+\kappa+\varepsilon)(b-t_0)}\Big) > -\rho,
\end{aligned}
$$

which contradicts the fact that $t_0 \in (a, b)$ and (a, b) is of type (Lo).
2. Suppose that $k_2 < 1/2$ and (a, b) is of type (Med), which by Equation (A.8) implies that

$$
M(t) = \Big(M(a) - \tfrac{1}{2}\Big)e^{-2\varepsilon(t-a)} + \tfrac{1}{2}, \qquad t \in [a, b]. \tag{4.2}
$$

With this, let us define

$$
\tilde{M} : [0, \infty) \to \mathbb{R}, \qquad t \mapsto \tilde{M}(t) := \tfrac{1}{2}\Big(1 - e^{-2\varepsilon(t-a)}\Big)
$$

and conclude that

$$
M(t) - \tilde{M}(t) = M(a)e^{-2\varepsilon(t-a)} \geq 0, \qquad t \in [a, b].
$$

Since $k_2 < 1/2$ by assumption, there exists $\lambda > 0$ sufficiently small with $k_2 + \lambda < 1/2$. Moreover, as $\tilde{M}$ is strictly increasing and $\tilde{M}(t) \to 1/2$ as $t \to \infty$, there exists a constant $L_1 > 0$ which does not depend on a, b, T such that $\tilde{M}(L_1 + a) > k_2 + \lambda$. Now define

$$
\eta := -\frac{g\big(k_2 + \lambda\big)}{\beta + 2\varepsilon} - \rho.
$$

Since $g(k_2 + \lambda) < g(k_2)$, it follows from the definition of k_2 in Equation (2.6) that

$$
\eta = -\frac{g\big(k_2 + \lambda\big)}{\beta + 2\varepsilon} - \rho > -\frac{g\big(k_2\big)}{\beta + 2\varepsilon} - \rho = -\frac{-(\beta + 2\varepsilon)\rho}{\beta + 2\varepsilon} - \rho = 0.
$$

Now there exists $L_2 > 0$ which does not depend on a, b, T such that

$$
(2\rho + \eta)e^{-(\beta+2\varepsilon)L_2} < \eta, \tag{4.3}
$$

and we define $L := L_1 + L_2$. As before, we argue by contradiction and assume that $b - a > L$. In that case, we have $a + L_1 < a + L < b$. The choice of L_1 therefore gives

$$
M(L_1 + a) \geq \tilde{M}(L_1 + a) > k_2 + \lambda.
$$

Hence, by continuity of M, there exists $t_0 \in (a, a + L_1)$ with $M(t_0) > k_2 + \lambda$. Analogously, we obtain

$$
M(b) \geq \tilde{M}(b) = \tilde{M}(b - a + a) > \tilde{M}(L_1 + a) > k_2 + \lambda.
$$

Hence, by monotonicity of g and since $\lambda > 0$, we have

$$\max\Big\{g\big(M(b)\big), g\big(M(t_0)\big)\Big\} < g(k_2 + \lambda) < g(k_2).$$

Since $g(M(t_0)) < g(k_2 + \lambda)$, it follows from the definition of η that

$$-\frac{g\big(M(t_0)\big)}{\beta + 2\varepsilon} > -\frac{g\big(k_2 + \lambda\big)}{\beta + 2\varepsilon} = \eta + \rho$$

and similarly, as $g(M(b)) < g(k_2 + \lambda)$,

$$-\frac{g\big(M(b)\big)}{\beta + 2\varepsilon} > -\frac{g\big(k_2 + \lambda\big)}{\beta + 2\varepsilon} = \eta + \rho.$$

We now distinguish the two cases $M(a) < 1/2$ and $M(a) \geq 1/2$. In the first case, it follows from Equation (4.2) that M is strictly increasing on (a, b) and hence $g(M)$ is strictly decreasing on (a, b). Using the explicit formula for W on type (Med) intervals given in Equation (A.10), we see that

$$\begin{aligned}
W(t_0) &= W(b)e^{-(\beta+2\varepsilon)(b-t_0)} - \int_{t_0}^{b} e^{(\beta+2\varepsilon)(t_0-s)} g\big(M(s)\big)\,\mathrm{d}s \\
&> W(b)e^{-(\beta+2\varepsilon)(b-t_0)} - g\big(M(t_0)\big)\int_{t_0}^{b} e^{(\beta+2\varepsilon)(t_0-s)}\,\mathrm{d}s \\
&= W(b)e^{-(\beta+2\varepsilon)(b-t_0)} - \frac{g\big(M(t_0)\big)}{\beta + 2\varepsilon}\Big(1 - e^{-(\beta+2\varepsilon)(b-t_0)}\Big) \\
&> W(b)e^{-(\beta+2\varepsilon)(b-t_0)} + (\rho + \eta)\Big(1 - e^{-(\beta+2\varepsilon)(b-t_0)}\Big) \\
&= \rho + \eta + \Big(W(b) - \rho - \eta\Big)e^{-(\beta+2\varepsilon)(b-t_0)}.
\end{aligned}$$

In the second case, that is, when $M(a) \geq 1/2$, we see from Equation (4.2) that $M \geq 1/2$ everywhere on $[a, b]$ so that M is decreasing and $g(M)$ is increasing on (a, b). Thus, we have

$$\begin{aligned}
W(t_0) &= W(b)e^{-(\beta+2\varepsilon)(b-t_0)} - \int_{t_0}^{b} e^{(\beta+2\varepsilon)(t_0-s)} g\big(M(s)\big)\,\mathrm{d}s \\
&\geq W(b)e^{-(\beta+2\varepsilon)(b-t_0)} - g\big(M(b)\big)\int_{t_0}^{b} e^{(\beta+2\varepsilon)(t_0-s)}\,\mathrm{d}s \\
&= W(b)e^{-(\beta+2\varepsilon)(b-t_0)} - \frac{g\big(M(b)\big)}{\beta + 2\varepsilon}\Big(1 - e^{-(\beta+2\varepsilon)(b-t_0)}\Big) \\
&> W(b)e^{-(\beta+2\varepsilon)(b-t_0)} + (\rho + \eta)\Big(1 - e^{-(\beta+2\varepsilon)(b-t_0)}\Big) \\
&= \rho + \eta + \Big(W(b) - \rho - \eta\Big)e^{-(\beta+2\varepsilon)(b-t_0)},
\end{aligned}$$

which is the same expression as for $M(a) < 1/2$. Since $t_0 < a + L_1$ and $b - a > L = L_1 + L_2$, we obtain

$$b - t_0 > b - a - L_1 > L - L_1 = L_2.$$

Using this and the fact that $W(b) \geq -\rho$ we conclude that

$$W(t_0) > \rho + \eta - \big(2\rho + \eta\big)e^{-(\beta+2\varepsilon)(b-t_0)} > \rho + \eta - \big(2\rho + \eta\big)e^{-(\beta+2\varepsilon)L_2} > \rho,$$

where we have used (4.3) for the last inequality. But this is the desired contradiction since (a, b) is an interval of type (Med), implying that $W(t_0) < \rho$. $\square$

We can now conclude that type (Lo) intervals are bounded independently of T. To see this, we recall that we have two scenarios in which intervals of type (Lo) appear, namely

$$(\mathrm{Lo}) \to (\mathrm{Med}) \qquad \text{and} \qquad (\mathrm{Lo}) \to (\mathrm{Med}) \to (\mathrm{Hi}) \to (\mathrm{Med}).$$

These two scenarios have in common that the type (Lo) interval is always followed up by a type (Med) interval and hence, denoting the type (Lo) interval by (a, b), we have $W(b) = -\rho$. This implies that Proposition 4.1.1 is applicable and hence any type (Lo) interval is bounded by a universal constant $L > 0$ which does not depend on T.

The case of type (Med) intervals is even more straightforward. By Proposition 4.1.2, any type (Med) interval is bounded by a universal constant $L > 0$ which is independent of T. From this, we conclude that the scenarios

$$(\mathrm{Med}) \qquad \text{and} \qquad (\mathrm{Lo}) \to (\mathrm{Med})$$

are impossible for large T. Moreover, in the remaining scenarios

$$(\mathrm{Hi}) \to (\mathrm{Med}) \quad \text{and} \quad (\mathrm{Med}) \to (\mathrm{Hi}) \to (\mathrm{Med}) \quad \text{and} \quad (\mathrm{Lo}) \to (\mathrm{Med}) \to (\mathrm{Hi}) \to (\mathrm{Med})$$

the intervals of type (Lo) and (Med) are universally bounded, meaning that the intervals of type (Hi) become arbitrarily large as $T \to \infty$. With this, we can show that the dynamic equilibria converge to the unique stationary equilibrium as $T \to \infty$, thus establishing the turnpike property.

Theorem 4.2 (Convergence of Dynamic Equilibria in Case (E1))**.** *Suppose that $k_1 < \varepsilon/(\kappa + \varepsilon)$ and $k_2 < 1/2$. Assume moreover that the terminal condition satisfies*

$$-\rho < G(m) < \rho, \qquad m \in [0, 1].$$

For any $T > 0$, let (M^T, W^T) solve the equilibrium system (2.2) – (2.3) *on $[0, T]$. Then there exists $L > 0$ such that, for all $T > 2L$ and all initial values $M^T(0) \in [0, 1]$, we have*

$$W^T(t) \geq \rho \qquad \textit{for all } t \in (L, T - L).$$

In particular, on the interval $(L, T - L)$, the action $h = (stay, switch)$ is optimal. Finally, for the every $\alpha \in (0, 1)$, it holds that

$$\lim_{T\to\infty} M^T\big(\alpha T\big) = \tfrac{\kappa}{\kappa+\varepsilon},$$

i.e. the dynamic equilibrium converges to the unique stationary equilibrium as $T \to \infty$. $\diamond$

Proof. From the discussion above, it follows that there exists $L > 0$ such that the interval $(L, T - L)$ is contained in an interval of type (Hi) for all $T > 2L$. If we denote this type (Hi) interval by (a_T, b_T), we have $a_T \leq L$ and it follows from $\alpha T - a_T \to \infty$ and uniform boundedness of M^T that

$$M^T(\alpha T) = \Big(M^T(a_T) - \tfrac{\kappa}{\kappa+\varepsilon}\Big)e^{-(\kappa+\varepsilon)(\alpha T - a_T)} + \tfrac{\kappa}{\kappa+\varepsilon} \to \tfrac{\kappa}{\kappa+\varepsilon} \qquad \text{as } T \to \infty,$$

where we have used the closed-form expression for M given in Equation (A.11). □

5. The Non-Unique Stationary Equilibrium Case (E2)

Let us now move on to the case (E2) in which the stationary equilibrium is no longer unique. We remind ourselves that in (E2) we assume that $k_1 < \varepsilon/(\kappa+\varepsilon)$ and $1/2 < k_2 < \kappa/(\kappa+\varepsilon)$. We furthermore recall that in this situation there are two deterministic equilibria at $M = 1/2$ and at $M = \kappa/(\kappa+\varepsilon)$ as well as one randomized equilibrium at $M = k_2$. Finally, according to Theorem A.4, we can no longer guarantee that $\{W \in \{-\rho, \rho\}\}$ has Lebesgue measure zero. In fact, while $k_1 < \varepsilon/(\kappa+\varepsilon)$ implies that $\{W = -\rho\}$ has measure zero and hence there are no intervals of type (Eq^-), the choice of k_2 implies that it is possible that $\{W = \rho\}$ has positive measure and there are possibly intervals of type (Eq^+).

5.1. Structure of the Solution (M, W) and Long-Run Behavior

Similarly to the case with a unique stationary equilibrium, the first step is to analyze the structure of the solution of the equilibrium system. As before, we could use a backward induction to determine all possible scenarios. In the present case (E2) the structure is, however, more sophisticated as the solution may exhibit cyclic behavior as we shall see shortly. In particular, since there is nothing to be gained by restricting the values of W at time T to get the backward induction started, we now allow for an arbitrary continuous (and hence bounded) terminal condition G. Regarding the structure of the solution of the equilibrium system, we make the following general observations.

- ▷ If there is a non-empty interval $(a, b) \subset [0, T]$ of type (Lo) with $b < T$, then $a = 0$. Indeed, in this case we have

$$M(t) = \Big(M(b) - \tfrac{\varepsilon}{\kappa+\varepsilon}\Big)e^{(\kappa+\varepsilon)(b-t)} + \tfrac{\varepsilon}{\kappa+\varepsilon}, \qquad t \in [a, b]$$

  by Equation (A.6). Now $W(b) = -\rho$ and $b > 0$ imply that $M(b) \leq k_1 < \varepsilon/(\kappa+\varepsilon)$ by Lemma 3.3.2. Thus M is strictly increasing on $[a, b]$ and hence $M(a) < M(b) \leq k_1$. Assuming now that $a > 0$, Lemma 3.1.1 implies $M(a) \geq k_1$, which is an immediate contradiction.
- ▷ It is not possible to have a chain of intervals of the form $(\mathrm{Eq}^+) \to (\mathrm{Med}) \to (\mathrm{Eq}^+)$. Indeed, let $a, b, c, d \in [0, T]$ with $a < b < c < d$ and suppose that the intervals (a, b) and (c, d) are of type (Eq^+) whereas (b, c) is of type (Med). Then $M(b) = M(c) = k_2$ by

the properties of intervals of type (Eq^+). On the other hand, on type (Med) intervals, M takes the form

$$M(t) = \Big(M(c) - \tfrac{1}{2}\Big)e^{2\varepsilon(c-t)} + \tfrac{1}{2}, \qquad t \in [b,c],$$

by Equation (A.9). Since $M(c) = k_2 > 1/2$, we see that M is strictly decreasing on $[b,c]$ which contradicts $M(b) = M(c) = k_2$.

▷ By the same argument, there is no chain of intervals of the form $(\mathrm{Eq}^+) \to (\mathrm{Hi}) \to (\mathrm{Eq}^+)$.

From these arguments we see that type (Lo) intervals are either of the form $(0,b)$ for some $b \in (0,T]$ or of the form (a,T) for some $a \in [0,T)$. In the former case, i.e. if $(0,b)$ is an interval of type (Lo), Proposition 4.1.1 is still applicable and it follows that b is bounded by a constant which does not depend on T. We now establish a similar result for the other case.

Lemma 5.1. *For $a \in [0,T)$, suppose that the interval (a,T) is of type (Lo). Then there exists a constant $L > 0$ which does not depend on T such that $T - a \leq L$.* ◇

Proof. Since $k_1 < \varepsilon/(\kappa+\varepsilon)$ and g is strictly decreasing, it follows that $g(k_1) > g(\varepsilon/(\kappa+\varepsilon))$. By continuity of g, it therefore follows that there exist $\lambda \in (0, \kappa/(\kappa+\epsilon))$ and $\eta > 0$ such that

$$g(m) \leq g(k_1) - \eta = (\beta + 2\varepsilon)\rho - \eta \quad \text{for all } m \in \Big[\tfrac{\varepsilon}{\kappa+\varepsilon} - \lambda, \tfrac{\varepsilon}{\kappa+\varepsilon} + \lambda\Big]. \tag{5.1}$$

Now choose $L_1, L_2 > 0$ sufficiently large such that

$$\tfrac{\kappa}{\kappa+\varepsilon}e^{-(\kappa+\varepsilon)L_1} = \lambda \qquad \text{and} \qquad -\rho - \eta L_2 < \min_{m\in[0,1]} G(m).$$

We claim that $T - a < L := L_1 + L_2$. To see this, we argue by contradiction and assume that $T - a \geq L$. We first show that

$$\Big|M(t) - \tfrac{\varepsilon}{\kappa+\varepsilon}\Big| \leq \lambda, \qquad t \in [a + L_1, T].$$

Indeed, since (a,T) is of type (Lo), we find that

$$M(t) = \Big(M(a) - \tfrac{\varepsilon}{\kappa+\varepsilon}\Big)e^{-(\kappa+\varepsilon)(t-a)} + \tfrac{\varepsilon}{\kappa+\varepsilon}, \qquad t \in [a,T]$$

by Equation (A.5). But then, for any $t \in [a + L_1, T]$,

$$\Big|M(t) - \tfrac{\varepsilon}{\kappa+\varepsilon}\Big| = \Big|M(a) - \tfrac{\varepsilon}{\kappa+\varepsilon}\Big|e^{-(\kappa+\varepsilon)(t-a)} \leq \tfrac{\kappa}{\kappa+\varepsilon}e^{-(\kappa+\varepsilon)(t-a)} \leq \tfrac{\kappa}{\kappa+\varepsilon}e^{-(\kappa+\varepsilon)L_1} = \lambda$$

by the choice of L_1. In particular, it follows from (5.1) that

$$g\big(M(t)\big) \leq (\beta + 2\varepsilon)\rho - \eta, \qquad t \in [a + L_1, T].$$

But then, again using that (a,T) is of type (Lo), we find that

$$\dot W(t) = g\big(M(t)\big) + C + (\beta + \kappa + \varepsilon)W(t)$$

$$\leq (\beta + 2\varepsilon)\rho - \eta + (\kappa - \varepsilon)\rho - (\beta + \kappa + \varepsilon)\rho = -\eta, \qquad t \in [a + L_1, T],$$

where we have used $W < -\rho$ on $[a, T]$ and $C = (\kappa - \varepsilon)\rho$. Since $T - a \geq L_1 + L_2$ by assumption, or, equivalently $T - L_2 \geq a + L_1$, the above estimate on $\dot{W}(t)$ holds in particular for all $t \in [T - L_2, T]$. But then

$$G\big(M(T)\big) = W(T) = W(T - L_2) + \int_{T-L_2}^{T} \dot{W}(t)\,\mathrm{d}t \leq -\rho - \eta L_2 < \min_{m \in [0,1]} G(m)$$

by the choice of L_2 and hence we have arrived at the desired contradiction. □

With the previous result at hand, we conclude that type (Lo) intervals play no role in the long-run behavior of (M, W) as they are universally bounded. Other than that, in full generality, the possible behavior of (M, W) cannot be restricted any further. In particular, it is possible that the solution exhibits cyclic behavior of the form

$$\begin{gathered}
\ldots(\mathrm{Med}) \to (\mathrm{Hi}) \to (\mathrm{Med})\ldots \\
\ldots(\mathrm{Med}) \to (\mathrm{Eq}^+) \to (\mathrm{Hi}) \to (\mathrm{Med})\ldots \\
\ldots(\mathrm{Med}) \to (\mathrm{Hi}) \to (\mathrm{Eq}^+) \to (\mathrm{Med})\ldots \\
\ldots(\mathrm{Med}) \to (\mathrm{Eq}^+) \to (\mathrm{Med})\ldots \\
\ldots(\mathrm{Hi}) \to (\mathrm{Eq}^+) \to (\mathrm{Hi})\ldots
\end{gathered}$$

Moreover, these cycles can be combined as long as no sequences of the form $(\mathrm{Eq}^+) \to (\mathrm{Med}) \to (\mathrm{Eq}^+)$ or $(\mathrm{Eq}^+) \to (\mathrm{Hi}) \to (\mathrm{Eq}^+)$ arise. We study the existence and properties of cycles in much more detail in the next subsection. Before getting there, however, let us first establish the main convergence result for the case (E2).

Theorem 5.2 (Convergence of Dynamic Equilibria in Case (E2))**.** *Suppose that $k_1 < \varepsilon/(\kappa + \varepsilon)$ and $1/2 < k_2 < \kappa/(\kappa + \varepsilon)$. For any $T > 0$, let (M^T, W^T) solve the equilibrium system* (2.2) – (2.3) *on $[0, T]$. Then at least one of the following statements holds.*

1. *There exists $\{T_n\}_{n\in\mathbb{N}} \subset (0, \infty)$ with $\lim_{n\to\infty} T_n = \infty$, such that, for each $n \in \mathbb{N}$, there is an interval $(a_n, b_n) \subset [0, T_n]$ of type* (Med) *and it holds that*

$$\lim_{n\to\infty} \big[b_n - a_n\big] = \infty \qquad \textit{and} \qquad \lim_{n\to\infty} M^{T_n}\big((a_n + b_n)/2\big) = \tfrac{1}{2},$$

   that is, the dynamic equilibrium converges to the stationary equilibrium at $M = 1/2$ as $n \to \infty$.

2. *There exists $\{T_n\}_{n\in\mathbb{N}} \subset (0, \infty)$ with $\lim_{n\to\infty} T_n = \infty$, such that, for each $n \in \mathbb{N}$, there is an interval $(a_n, b_n) \subset [0, T_n]$ of type* (Hi) *and it holds that*

$$\lim_{n\to\infty} \big[b_n - a_n\big] = \infty \qquad \textit{and} \qquad \lim_{n\to\infty} M^{T_n}\big((a_n + b_n)/2\big) = \tfrac{\kappa}{\kappa+\varepsilon},$$

   that is, the dynamic equilibrium converges to the stationary equilibrium at $M = \kappa/(\kappa + \varepsilon)$ as $n \to \infty$.

3. *There exists $\{T_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ with $\lim_{n\to\infty} T_n = \infty$, such that, for each $n \in \mathbb{N}$, there is an interval $(a_n, b_n) \subset [0, T_n]$ of type (Eq^+) and it holds that*

$$\lim_{n\to\infty} \big[b_n - a_n\big] = \infty \qquad \textit{and} \qquad \big(M^{T_n}(t), W^{T_n}(t)\big) = (k_2, \rho),$$

that is, the dynamic equilibrium spends an infinite amount of time in the randomized equilibrium at $M = k_2$ as $n \to \infty$.

4. *There exists $\{T_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ with $\lim_{n\to\infty} T_n = \infty$, such that, for each $n \in \mathbb{N}$, all subintervals of $[0, T_n]$ of type (Lo), (Med), (Hi), and (Eq^+) remain bounded. In that case, for each $n \in \mathbb{N}$, the dynamic equilibrium cycles around the randomized stationary equilibrium (k_2, ρ).* ◇

Proof. Since intervals of type (Lo) are universally bounded, only intervals of type (Med), (Hi), and (Eq^+) can become unbounded in the limit. As such, the four cases given in the theorem cover all possible limit behaviors of the equilibrium system, which is to say that at least one of the cases has to occur. We now look at all four cases individually to establish the remaining claims. To begin with, let $\{T_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ with $\lim_{n\to\infty} T_n = \infty$, such that, for each $n \in \mathbb{N}$, there is an interval $(a_n, b_n) \subset [0, T_n]$ of type (Med) such that $b_n - a_n \to \infty$ as $n \to \infty$. Using the closed-form expression for M on intervals of type (Med) given in Equation (A.8), it follows that

$$\lim_{n\to\infty} M^{T_n}\big((a_n + b_n)/2\big) = \lim_{n\to\infty} \Big(M(a_n) - \tfrac{1}{2}\Big) e^{-\varepsilon(b_n - a_n)} + \tfrac{1}{2} = \tfrac{1}{2}.$$

This concludes the first case. The convergence in the second case is argued for analogously. In the third case, there is nothing to show. Finally, the cyclic behavior in the fourth case is established in Lemma 5.4 in the next subsection. □

While the previous theorem gives a full account on the behavior of the equilibrium system in the limit as $T \to \infty$, we have not yet shown that all four cases may actually be observed. Regarding the fourth case, we postpone the proof of existence of cycles and a detailed study of the properties of cycles to the next subsection. Regarding the other three cases, it is simple to show that these occur. Indeed, if we assume that

$$M_0 = \frac{1}{2} \qquad \text{and} \qquad G \equiv W_T := -\frac{g(1/2)}{\beta + 2\varepsilon},$$

it is straightforward to see that there is a constant solution of the equilibrium system taking the value (M_0, W_T), and the entire time interval $[0, T]$ is of type (Med). Similarly, if

$$M_0 = \frac{\kappa}{\kappa + \varepsilon} \qquad \text{and} \qquad G \equiv W_T := -\frac{g\big(\kappa/(\kappa+\varepsilon)\big) - C}{\beta + \kappa + \varepsilon},$$

there is a constant solution taking the value (M_0, W_T) and $[0, T]$ is of type (Hi). Finally, if

$$M_0 = k_2 \qquad \text{and} \qquad G \equiv \rho,$$

it follows that there is a constant solution taking the value (k_2, ρ), meaning that $[0, T]$ is of type (Eq^+). As such, all three cases are possible and it remains to study the possibility of cyclic behavior of the equilibrium system.

5.2. Analysis of the Cycles in the Equilibrium System

We are left with establishing the existence of cycles in case (E2), which then implies that Theorem 5.2 cannot be improved in the sense that all four cases actually occur. In what follows, we shall see that the dynamical system (M, W) indeed admits cycles which all revolve around the randomized equilibrium located at (k_2, ρ). Figure 1 below shows a streamline plot of the system in a neighborhood of this equilibrium and showing strong numerical evidence for the existence of cycles.

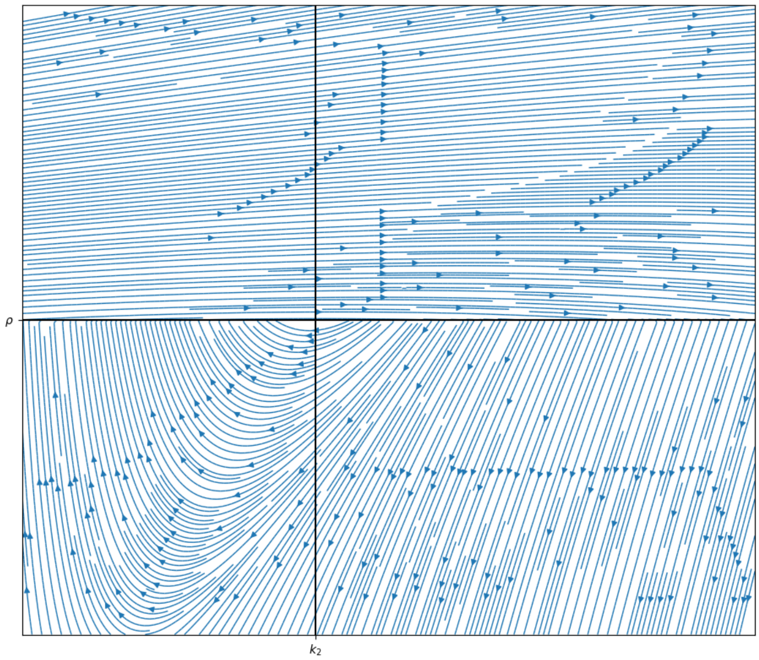

Figure 1: A streamline plot of the equilibrium system in the (M, W)-plane indicating that the dynamic equilibrium circles around the randomized stationary equilibrium at (k_2, ρ).

To facilitate the study of the cyclic behavior of the solution of the equilibrium system, let us first agree on a precise naming convention. For now, we restrict our attention to cycles which do not involve intervals of type (Eq^+).

Definition 5.3. *Let (a, b) be a non-empty open subinterval of $[0, T]$.*

1. *We say that (a, b) is a halfcycle of type* (Hi) *if $W(a) = W(b) = \rho$ and $W > \rho$ on (a, b).*
2. *We say that (a, b) is a halfcycle of type* (Med) *if $W(a) = W(b) = \rho$ and $W < \rho$ on (a, b).*

With this, if $c \in (b, T)$, we say that (a, b, c) is a cycle if (a, b) is a halfcycle of type (Hi) *and (b, c) is a halfcycle of type* (Med). ◇

We start our discussion of the cyclic behavior by collecting some properties of halfcycles.

Lemma 5.4. *Let (a, b) be a non-empty open subinterval of $[0, T]$.*

1. *If (a, b) is a halfcycle of type* (Hi)*, the following statements hold.*
   (i) M is strictly increasing on $[a, b]$ and satisfies $M(a) \le k_2 \le M(b) < \kappa/(\kappa + \varepsilon)$.
   (ii) There is a unique $t_0 \in (a, b)$ such that $\dot W(t_0) = 0$. Moreover, we have $\dot W(t) > 0$ for all $t \in (a, t_0)$, $\dot W(t) < 0$ for all $t \in (t_0, b)$, and

$$W(t_0) = -\frac{g\big(M(t_0)\big) - C}{\beta + \kappa + \varepsilon} \qquad \textit{and} \qquad M(t_0) > k_2.$$

2. *If (a, b) is a halfcycle of type* (Med)*, the following statements hold.*
   (i) M is strictly decreasing on $[a, b]$ and satisfies $M(a) \ge k_2 \ge M(b) > 1/2$.
   (ii) There is a unique $t_0 \in (a, b)$ such that $\dot W(t_0) = 0$. Moreover, we have $\dot W(t) < 0$ for all $t \in (a, t_0)$, $\dot W(t) > 0$ for all $t \in (t_0, b)$, and

$$W(t_0) = -\frac{g\big(M(t_0)\big)}{\beta + 2\varepsilon} \qquad \textit{and} \qquad M(t_0) < k_2. \qquad \diamond$$

Proof. We only prove the first statement on halfcycles of type (Hi) since the second statement follows analogously. We proceed in three steps.

Step 1: We prove (i). In order to show that $M(a) \le k_2$, we first observe that $W > \rho$ on (a, b) and $W(a) = \rho$ imply $\dot W(a) \ge 0$. But then

$$0 \le \dot W(a) = g\big(M(a)\big) - C + (\beta + \kappa + \varepsilon)W(a) = g\big(M(a)\big) + (\beta + 2\varepsilon)\rho,$$

where we have used $W(a) = \rho$ and $C = (\kappa - \varepsilon)\rho$. Rearranging terms and using the definition of k_2 in Equation (2.6) thus yields

$$g\big(M(a)\big) \ge -(\beta + 2\varepsilon)\rho = g(k_2),$$

and we conclude that $M(a) \le k_2$ since g is decreasing. The inequality $M(b) \ge k_2$ follows from an analogous argument. Next, since (a, b) is of type (Hi), we have

$$M(t) = \Big(M(a) - \tfrac{\kappa}{\kappa+\varepsilon}\Big) e^{-(\kappa+\varepsilon)(t-a)} + \tfrac{\kappa}{\kappa+\varepsilon}, \qquad t \in [a, b]$$

by Equation (A.11). Since $M(a) \le k_2 < \kappa/(\kappa + \varepsilon)$, we see that M is strictly increasing on $[a, b]$ and hence $M(a) < M(b)$. Similarly, we have

$$M(b) = \Big(M(a) - \tfrac{\kappa}{\kappa+\varepsilon}\Big) e^{-(\kappa+\varepsilon)(b-a)} + \tfrac{\kappa}{\kappa+\varepsilon} < \tfrac{\kappa}{\kappa+\varepsilon},$$

which concludes part (i).

Step 2: Preparations for (ii). By construction, M and W satisfy

$$\dot M(t) = \kappa - (\kappa + \varepsilon)M(t), \qquad\qquad t \in (a, b),$$

$$\dot{W}(t) = g\big(M(t)\big) - C + (\beta + \kappa + \varepsilon)W(t), \qquad t \in (a,b),$$

with $M(a) < \kappa/(\kappa+\varepsilon)$. We proceed to show that if there exists $t_0 \in (a, T)$ with $\dot{W}(t_0) = 0$, then W is strictly decreasing on $[t_0, b]$. For this, we first note that the second-order derivative satisfies

$$\ddot{W}(t) = \dot{g}\big(M(t)\big)\dot{M}(t) + (\beta + \kappa + \varepsilon)\dot{W}(t), \qquad t \in (a,b).$$

Note that, according to Equation (A.11), the function M is given explicitly as

$$M(t) = \Big(M(a) - \tfrac{\kappa}{\kappa+\varepsilon}\Big)e^{-(\kappa+\varepsilon)(t-a)} + \tfrac{\kappa}{\kappa+\varepsilon}, \qquad t \in [a,b],$$

from which we conclude that M is strictly increasing since $M(a) < \kappa/(\kappa+\varepsilon)$ by part (i). In particular, we get $\dot{M}(t) > 0$ for all $t \in (a,b)$ and since g is strictly decreasing, it follows that

$$\ddot{W}(t) < (\beta + \kappa + \varepsilon)\dot{W}(t), \qquad t \in (a,b).$$

Now $\dot{W}(t_0) = 0$, so W is strictly concave on $[t_0, b]$. Together with $\dot{W}(t_0) = 0$ this implies that W is strictly decreasing on $[t_0, b]$ as claimed.

Step 3: We are now ready to prove (ii). By step 1, we know that M is strictly increasing on $[a, b]$. Since $W(a) = W(b) = \rho$ and $W(t) > \rho$ for all $t \in (a, b)$, it follows from continuity that W admits a maximum on (a, b). Let t_0 be the smallest number in (a, b) at which W attains a local maximum. Since $\dot{W}(t_0) = 0$, it follows from step 2 that W is strictly decreasing on $[t_0, b]$ so that t_0 is in fact the unique maximum of W on $[a, b]$. Next, we note that

$$0 = \dot{W}(t_0) = g\big(M(t_0)\big) - C + (\beta + \kappa + \varepsilon)W(t_0), \quad \text{that is,} \quad W(t_0) = -\frac{g\big(M(t_0)\big) - C}{\beta + \kappa + \varepsilon}$$

as claimed. Finally, after rearranging terms and using $W(t_0) > \rho$ as well as $C = (\kappa - \varepsilon)\rho$, we conclude that

$$g\big(M(t_0)\big) = C - (\beta + \kappa + \varepsilon)W(t_0) < (\kappa - \varepsilon)\rho - (\beta + \kappa + \varepsilon)\rho = -(\beta + 2\varepsilon)\rho = g(k_2),$$

implying that $M(t_0) > k_2$ since g is strictly decreasing. □

The previous lemma implies that for any cycle (a, b, c) as in Definition 5.3 we have

$$M(a) \le k_2 \qquad \text{and} \qquad k_2 \le M(b) < \tfrac{\kappa}{\kappa+\varepsilon} \qquad \text{and} \qquad \tfrac{1}{2} < M(c) \le k_2,$$

with $M(a) < M(b)$ and $M(b) > M(c)$. Thus any cycle revolves around the randomized equilibrium k_2 and may possibly touch it. In fact, according to the discussion in the previous subsection, there are also scenarios in which the trajectory of (M, W) spends a positive amount of time in (k_2, ρ) in between halfcycles.

The extremal values of a halfcycle (a, b) in the the M-direction are precisely $M(a)$ and $M(b)$. In the W-direction, the extremal values are ρ and $W(t_0)$, where $t_0 \in (a, b)$ is the unique point with $\dot{W}(t_0) = 0$. We subsequently refer to a as the starting point and $M(a)$ as the *starting value* of the halfcycle. Similarly, we refer to t_0 as the inflection point and $W(t_0)$ as the *inflection* *value* of the halfcycle. In what follows, we analyze which points in $m \in [0,1]$ can be starting values of a halfcycle and which points $w \in \mathbb{R}$ can be inflection values of a halfcycle.

To simplify the exposition, we observe that we may subsequently abandon the terminal condition and work on more general time index sets. More precisely, given a closed interval A with interior denoted by I, we may consider functions

$$\mathbf{m} : A \to [0,1] \qquad \text{and} \qquad \mathbf{w} : A \to \mathbb{R}$$

such that $\mathbf{m}$ is absolutely continuous, $\mathbf{w}$ is continuously differentiable, and

$$\dot{\mathbf{m}}(t) = F_{\to}\big(\mathbf{m}(t), \mathbf{w}(t)\big) \text{ a.e.} \qquad \text{and} \qquad \dot{\mathbf{w}}(t) = F_{\leftarrow}\big(\mathbf{m}(t), \mathbf{w}(t)\big), \qquad t \in I.$$

Any such pair $(\mathbf{m}, \mathbf{w})$ is referred to as a *trajectory* of the equilibrium system. If a trajectory has a halfcycle (a,b), it follows that there is a solution (M,W) of the equilibrium system with the same halfcycle. Indeed, we may choose $t_0 \in \mathbb{R}$ and $T > 0$ such that $(a,b) \subset [t_0, t_0 + T] \subseteq A$ and define a solution of the equilibrium system by setting $(M,W) := (\mathbf{m}(\cdot - t_0), \mathbf{w}(\cdot - t_0))$ and choosing G such that $G(M(T)) \equiv \mathbf{w}(t_0 + T)$. Then $(a - t_0, b - t_0)$ is a halfcycle of (M,W). With this, it follows that in order to show that the equilibrium system admits halfcycles, it is sufficient to show that there exist trajectories having halfcycles.

Lemma 5.5. *The following statements hold.*

1. *There exists a trajectory having a halfcycle of type* (Hi).
2. *There exists a trajectory having a halfcycle of type* (Med). $\diamond$

Proof. As usual we only prove the existence of a halfcycle of type (Hi) since the existence of a halfcycle of type (Med) follows analogously.

Step 1: In this step we construct a candidate $(\mathbf{m}, \mathbf{w})$ for the desired trajectory. For this, define

$$L := \tfrac{1}{\kappa+\varepsilon}\Big[\log\Big(\tfrac{\kappa}{\kappa+\varepsilon}\Big) - \log\Big(\tfrac{\kappa}{\kappa+\varepsilon} - k_1\Big)\Big] > 0$$

and choose $m \in (k_2, \kappa/(\kappa+\varepsilon))$ such that

$$g(k_2) - g(m) < L(\beta + \kappa + \varepsilon)\big(g(k_1) - g(k_2)\big), \tag{5.2}$$

which is possible since g is continuous and strictly decreasing. Now let $(\mathbf{m}, \mathbf{w})$ solve

$$\dot{\mathbf{m}}(t) = \kappa - (\kappa+\varepsilon)\mathbf{m}(t) \quad \text{and} \quad \dot{\mathbf{w}}(t) = g\big(\mathbf{m}(t)\big) - C + (\beta+\kappa+\varepsilon)\mathbf{w}(t), \qquad t \in \mathbb{R},$$

with boundary conditions

$$\mathbf{m}(0) = m \qquad \text{and} \qquad \mathbf{w}(0) = -\frac{g(m) - C}{\beta+\kappa+\varepsilon}.$$

Note that $m > k_2$ implies

$$\mathbf{w}(0) = -\frac{g(m) - C}{\beta+\kappa+\varepsilon} > -\frac{g(k_2) - C}{\beta+\kappa+\varepsilon} = \frac{(\beta+2\varepsilon)\rho + (\kappa-\varepsilon)\rho}{\beta+\kappa+\varepsilon} = \rho,$$

and we furthermore observe that

$$\dot{\mathbf{w}}(0) = g\big(\mathbf{m}(0)\big) - C + (\beta+\kappa+\varepsilon)\mathbf{w}(0) = g(m) - C - (\beta+\kappa+\varepsilon)\frac{g(m)-C}{\beta+\kappa+\varepsilon} = 0,$$

which is to say that $\mathbf{w}(0)$ is the inflection value. As in part (ii) of Lemma 5.4.1, we have $\dot{\mathbf{w}}(t) > 0$ for all $t < 0$ and $\dot{\mathbf{w}}(t) < 0$ for all $t > 0$.

Step 2: We now verify that $(\mathbf{m}, \mathbf{w})$ indeed has a halfcycle. For this, we need to show that there are $a < 0$ and $b > 0$ such that $\mathbf{w}(a) = \mathbf{w}(b) = \rho$, which means that $(\mathbf{m}, \mathbf{w})$ restricted to $[a, b]$ is a trajectory having a halfcycle (a, b). To that end, let us first define

$$t_0 := \tfrac{1}{\kappa+\varepsilon}\Big[\log\Big(\tfrac{\kappa}{\kappa+\varepsilon} - m\Big) - \log\Big(\tfrac{\kappa}{\kappa+\varepsilon}\Big)\Big] < 0.$$

We proceed to show that $\mathbf{w}(t_0) \le \rho$, so that the intermediate value theorem implies the existence of $a \in [t_0, 0]$ with $\mathbf{w}(a) = \rho$. We argue by contradiction and assume that $\mathbf{w}(t_0) > \rho$. First, we observe that

$$t_0 + L = \tfrac{1}{\kappa+\varepsilon}\Big[\log\Big(\tfrac{\kappa}{\kappa+\varepsilon} - m\Big) - \log\Big(\tfrac{\kappa}{\kappa+\varepsilon} - k_1\Big)\Big] < 0$$

since $m > k_2 > k_1$. Next, we note that $\mathbf{m}$ is given explicitly as

$$\mathbf{m}(t) = \Big(\mathbf{m}(0) - \tfrac{\kappa}{\kappa+\varepsilon}\Big)e^{-(\kappa+\varepsilon)t} + \tfrac{\kappa}{\kappa+\varepsilon}, \qquad t \le 0;$$

see Equation (A.12). From this, it follows that $\mathbf{m}$ is strictly increasing on $(-\infty, 0]$ since it holds that $\mathbf{m}(0) = m < \kappa/(\kappa+\varepsilon)$. Moreover, we have

$$\mathbf{m}(t_0 + L) = \Big(m - \tfrac{\kappa}{\kappa+\varepsilon}\Big)e^{-(\kappa+\varepsilon)(t_0+L)} + \tfrac{\kappa}{\kappa+\varepsilon} = k_1.$$

In particular, strict monotonicity of $\mathbf{m}$ and $\mathbf{w}$ on $(-\infty, 0]$ imply that

$$\mathbf{m}(t) < k_1 \qquad \text{and} \qquad \mathbf{w}(t) > \mathbf{w}(t_0) > \rho, \qquad t \in (t_0, t_0 + L).$$

With this, using that g is strictly decreasing and $C = (\kappa - \varepsilon)\rho$, we obtain

$$\begin{aligned}\dot{\mathbf{w}}(t) &= g\big(\mathbf{m}(t)\big) - C + (\beta+\kappa+\varepsilon)\mathbf{w}(t) \\ &> g(k_1) - (\kappa-\varepsilon)\rho + (\beta+\kappa+\varepsilon)\rho \\ &= g(k_1) + (\beta+2\varepsilon)\rho = g(k_1) - g(k_2), \qquad\qquad t \in (t_0, t_0+L).\end{aligned}$$

Hence, since $\dot{\mathbf{w}} > 0$ on $(-\infty, 0)$,

$$\mathbf{w}(0) = \mathbf{w}(t_0) + \int_{t_0}^{0} \dot{\mathbf{w}}(t)\,\mathrm{d}t > \mathbf{w}(t_0) + \int_{t_0}^{t_0+L} \dot{\mathbf{w}}(t)\,\mathrm{d}t > \rho + L\big(g(k_1) - g(k_2)\big).$$

Recalling that m was chosen such that (5.2) holds, $g(k_2) = -(\beta+2\varepsilon)\rho$, and $C = (\kappa-\varepsilon)\rho$, it follows that

$$\mathbf{w}(0) > \rho + L\big(g(k_1) - g(k_2)\big) > \rho + \frac{g(k_2) - g(m)}{\beta+\kappa+\varepsilon} = \frac{C}{\beta+\kappa+\varepsilon} - \frac{g(m)}{\beta+\kappa+\varepsilon} = \mathbf{w}(0),$$

which is the desired contradiction. Hence $\mathbf{w}(t_0) \leq \rho$, implying the existence of $a \in [t_0, 0)$ with $\mathbf{w}(a) = \rho$.

We are left with proving the existence of $b > 0$ with $\mathbf{w}(b) = \rho$ to finish the proof. For this, let us choose $t_1 > 0$ such that

$$\mathbf{m}(t_1) = \tfrac{1}{2}\Big(m + \tfrac{\kappa}{\kappa+\varepsilon}\Big),$$

which is possible since $\mathbf{m}$ is strictly increasing, $\mathbf{m}(t) \to \kappa/(\kappa+\varepsilon)$ as $t \to \infty$, and

$$\mathbf{m}(0) = m < \tfrac{1}{2}\Big(m + \tfrac{\kappa}{\kappa+\varepsilon}\Big) < \tfrac{\kappa}{\kappa+\varepsilon}.$$

Since $\mathbf{w}$ is strictly decreasing on $[0, \infty)$, we obtain

$$\mathbf{w}(t) < \mathbf{w}(0) = -\frac{g(m) - C}{\beta + \kappa + \varepsilon}, \qquad t \geq 0.$$

With this and since $\mathbf{m}$ is strictly increasing on $[0, \infty)$ and g is strictly decreasing, we find that

$$\begin{aligned}
\dot{\mathbf{w}}(t) &= g\big(\mathbf{m}(t)\big) - C + (\beta + \kappa + \varepsilon)\mathbf{w}(t) \\
&< g\big(\mathbf{m}(t_1)\big) - C + (\beta + \kappa + \varepsilon)\Big(-\frac{g(m) - C}{\beta + \kappa + \varepsilon}\Big) \\
&= g\big(\mathbf{m}(t_1)\big) - g(m) < 0, \qquad\qquad t \geq t_1.
\end{aligned}$$

Moreover, again using that $\dot{\mathbf{w}}(t) < 0$ for all $t > 0$, we have

$$\begin{aligned}
\mathbf{w}(t) &= \mathbf{w}(0) + \int_0^t \dot{\mathbf{w}}(s)\,\mathrm{d}s \leq \mathbf{w}(0) + \int_{t_1}^t \dot{\mathbf{w}}(s)\,\mathrm{d}s \\
&< \mathbf{w}(0) + (t - t_1)\big(g\big(\mathbf{m}(t_1)\big) - g(m)\big), \qquad\qquad t > t_1.
\end{aligned}$$

Since the term in in brackets is strictly negative it follows that $W(t) < \rho$ for all $t > t_1$ sufficiently large. In particular, since $\mathbf{w}(0) > \rho$, the intermediate value theorem guarantees the existence of $b > 0$ such that $\mathbf{w}(b) = \rho$. □

Having established the existence of one halfcycle of each type, we now show that indeed infinitely many halfcycles of each type exists. In particular, we prove that the set of all starting values of a halfcycle as well as the set of all inflection values of a halfcycle are intervals.

Lemma 5.6. *Let $(\mathbf{m}, \mathbf{w})$ be a trajectory.*

1. *If $(\mathbf{m}, \mathbf{w})$ has a halfcycle of type (Hi) with starting value $m < k_2$, then for all $m' \in (m, k_2)$ there is a trajectory having a halfcycle of type (Hi) with starting value m'.*
2. *If $(\mathbf{m}, \mathbf{w})$ has a halfcycle of type (Hi) with inflection value $w > \rho$, then for all $w' \in (\rho, w)$ there is a trajectory having a halfcycle of type (Hi) with inflection value w'.*
3. *If $(\mathbf{m}, \mathbf{w})$ has a halfcycle of type (Med) with starting value $m > k_2$, then for all $m' \in (k_2, m)$ there is a trajectory having a halfcycle of type (Med) with starting value m'.*
4. *If $(\mathbf{m}, \mathbf{w})$ has a halfcycle of type (Med) with inflection value $w < \rho$, then for all $w' \in (w, \rho)$ there is a trajectory having a halfcycle of type (Med) with inflection value w'.* ◇

Proof. As usual, we only prove the statements on halfcycles of type (Hi) as the remaining statements can be argued for analogously. To fix notations, we subsequently denote the halfcycle of $(\mathbf{m}, \mathbf{w})$ by (a, b) and write $t_0 \in (a, b)$ for the inflection point. We also set $m := \mathbf{m}(a)$ and $w := \mathbf{w}(t_0)$ for the starting value and the inflection value, respectively.

Step 1: Assume that $m < k_2$ and let $m' \in (m, k_2)$. We construct a trajectory $(\mathbf{m}', \mathbf{w}')$ having a halfcycle with starting value m'. For this, let us denote by $\mathfrak{A}$ the area enclosed by the graph of $t \mapsto (\mathbf{m}(t), \mathbf{w}(t))$, $t \in [a, b]$, and the line connecting $(\mathbf{m}(a), \rho)$ and $(\mathbf{m}(b), \rho)$. We consider another trajectory $(\mathbf{m}', \mathbf{w}') : \mathbb{R} \to [0, 1] \times \mathbb{R}$ such that

$$\mathbf{m}'(0) = m' \qquad \text{and} \qquad \mathbf{w}'(0) = \rho.$$

Since $m' < k_2$, monotonicity of g and $C = (\kappa - \varepsilon)\rho$ show that

$$\dot{\mathbf{w}}'(0) = g(m') - C + (\beta + \kappa + \varepsilon)\mathbf{w}(0) > g(k_2) + (\beta + 2\varepsilon)\rho = 0,$$

that is, $(\mathbf{m}', \mathbf{w}')$ enters the interior of the area $\mathfrak{A}$ after time $t = 0$. Since the graphs of $(\mathbf{m}, \mathbf{w})$ and $(\mathbf{m}', \mathbf{w}')$ cannot intersect, this argument also shows that $(\mathbf{m}', \mathbf{w}')$ can only exit the are $\mathfrak{A}$ at points $(\tilde{m}, \rho)$ with $\tilde{m} \geq k_2$. As such, to conclude that $(\mathbf{m}', \mathbf{w}')$ has a halfcycle with starting value $\mathbf{m}'(0) = m'$, it is sufficient to show that $(\mathbf{m}', \mathbf{w}')$ eventually leaves the area $\mathfrak{A}$, which is to say that there exists $t > 0$ such that $\mathbf{w}'(t) = \rho$. We argue by contradiction and assume that $\mathbf{w}' > \rho$ on all of $(0, \infty)$. In particular, it follows that $\mathbf{m}'$ is given explicitly as

$$\mathbf{m}'(t) = \Big(\mathbf{m}(0) - \tfrac{\kappa}{\kappa+\varepsilon}\Big)e^{-(\kappa+\varepsilon)t} + \tfrac{\kappa}{\kappa+\varepsilon}, \qquad t \in [0, \infty);$$

see Equation (A.11). In particular, $\mathbf{m}'(t) \to \kappa/(\kappa + \varepsilon)$ as $t \to \infty$. But since any $(m, w) \in \mathfrak{A}$ satisfies $m < M(b) < \kappa/(\kappa + \varepsilon)$, this is the desired contradiction. Hence $(\mathbf{m}', \mathbf{w}')$ has a halfcycle of type (Hi) with starting value m'.

Step 2: Assume that $w > \rho$ and let $w' \in (\rho, w)$. We construct a trajectory $(\mathbf{m}', \mathbf{w}')$ having a halfcycle with inflection value w'. For this, let us denote by m, m' the unique points satisfying

$$w = -\frac{g(m) - C}{\beta + \kappa + \varepsilon} \qquad \text{and} \qquad w' = -\frac{g(m') - C}{\beta + \kappa + \varepsilon}.$$

Note that $\mathbf{m}(t_0) = m$, where t_0 is the point of inflection of the halfcycle (a, b) of $(\mathbf{m}, \mathbf{w})$. Indeed, since $\mathbf{w}(t_0) = w$ and $\dot{\mathbf{w}}(t_0) = 0$, we get

$$0 = \dot{\mathbf{w}}(t_0) = g\big(\mathbf{m}(t_0)\big) - C + (\beta + \kappa + \varepsilon)\mathbf{w}(t_0) = g\big(\mathbf{m}(t_0)\big) - C + (\beta + \kappa + \varepsilon)w,$$

from which we conclude that $\mathbf{m}(t_0) = m$ by strict monotonicity of g. In particular, we have $m < \kappa/(\kappa + \varepsilon)$. Next, we observe that since

$$-\frac{g(k_2) - C}{\beta + \kappa + \varepsilon} = \frac{(\beta + 2\varepsilon)\rho + (\kappa - \varepsilon)\rho}{\beta + \kappa + \varepsilon} = \rho,$$

it follows from $\rho < w' < w$ and the fact that g is strictly decreasing that $k_2 < m' < m$. Now consider a trajectory $(\mathbf{m}', \mathbf{w}')$ defined on $\mathbb{R}$ such that

$$\mathbf{m}'(a) = m' \qquad \text{and} \qquad \mathbf{w}'(a) = w'.$$

Since

$$\dot{\mathbf{w}}'(a) = g\big(\mathbf{m}'(a)\big) - C + (\beta + \kappa + \varepsilon)\mathbf{w}'(a) = g(m') - C + (\beta + \kappa + \varepsilon)w' = 0,$$

the inflection value of the halfcycle is w' as desired. To conclude, it therefore suffices to show that there exist $a' < a$ and $b' > a$ such that $\mathbf{w}'(a') = \mathbf{w}'(b') = \rho$, implying that (a', b') is a halfcycle of type (Hi). For this let us again consider the area $\mathfrak{A}$, which is the area enclosed by the graph of $t \mapsto (\mathbf{m}(t), \mathbf{w}(t))$, $t \in [a, b]$, and the line connecting $(\mathbf{m}(a), \rho)$ and $(\mathbf{m}(b), \rho)$. Since $\mathbf{m}(a) \leq k_2 < m'$ and $\mathbf{m}(t_0) = m > m'$, it follows that there exists $t \in (a, t_0)$ such that $\mathbf{m}(t) = m'$. As in part (ii) of Lemma 5.4.1, we have $\dot{\mathbf{w}}(t) > 0$. This implies

$$g(m') - C + (\beta + \kappa + \varepsilon)\mathbf{w}(t) = \dot{\mathbf{w}}(t) > 0 = g(m') - C - (\beta + \kappa + \varepsilon)\frac{g(m') - C}{\beta + \kappa + \varepsilon},$$

and we conclude that

$$\mathbf{w}(t) > -\frac{g(m') - C}{\beta + \kappa + \varepsilon} = w'.$$

In particular, we have argued that $(\mathbf{m}'(a), \mathbf{w}'(a)) = (m', w') \in \mathfrak{A}$. We argued before that the trajectory $(\mathbf{m}', \mathbf{w}')$ can leave the area $\mathfrak{A}$ only at points $(\tilde{m}, \rho)$ both forward in time and backward in time. We now prove that the trajectory will eventually leave $\mathfrak{A}$, which then proves the claim. Assume not, i.e. there is no $a' < a$ such that $\mathbf{w}'(a') = \rho$ or no $b' > a$ such that $\mathbf{w}'(b') = \rho$. Then $\mathbf{m}'$ is given explicitly as

$$\mathbf{m}'(t) = \Big(\mathbf{m}'(a) - \tfrac{\kappa}{\kappa+\varepsilon}\Big)e^{-(\kappa+\varepsilon)(t-a)} + \tfrac{\kappa}{\kappa+\varepsilon}, \qquad t \in \mathbb{R}$$

by Equation (A.11). In particular, since $\mathbf{m}'(a) = m' < m < \kappa/(\kappa + \varepsilon)$, it follows that $\mathbf{m}'$ is strictly increasing with

$$\lim_{t \downarrow -\infty} \mathbf{m}'(t) = -\infty \qquad \text{and} \qquad \lim_{t \uparrow \infty} \mathbf{m}'(t) = \tfrac{\kappa}{\kappa+\varepsilon}.$$

But since any $(m, w) \in \mathfrak{A}$ satisfies $0 \leq \mathbf{m}(a) \leq m \leq \mathbf{m}(b) < \kappa/(\kappa + \varepsilon)$ this leads to the desired contradiction. $\square$

In conclusion, we have argued that as soon as the time horizon T is sufficiently large, there are (uncountably many) scenarios in which cycles can be observed. In such scenarios, the solution of the dynamic equilibrium system cycles around the randomized equilibrium at k_2. This wraps up our discussion of the case (E2).

A. Existence of Solutions of the Equilibrium System

In this appendix, we construct a solution of the equilibrium system

$$\begin{aligned} \dot{M}(t) &= F_{\rightarrow}\big(M(t), W(t)\big) && \text{for a.e. } t \in [0, T], && M(0) = M_0, \\ \dot{W}(t) &= F_{\leftarrow}\big(M(t), W(t)\big) && \text{for all } t \in [0, T], && W(T) = G\big(M(T)\big), \end{aligned}$$

where we recall that

$$F_{\to}(m,w) := \begin{cases} \varepsilon - (\kappa+\varepsilon)m & \text{if } w \le -\rho, \\ \varepsilon - 2\varepsilon m & \text{if } w \in (-\rho,\rho), \\ \kappa - (\kappa+\varepsilon)m & \text{if } w \ge \rho, \end{cases} \qquad (m,w) \in [0,1]\times\mathbb{R},$$

$$F_{\leftarrow}(m,w) := \begin{cases} g(m) + C + (\beta+\kappa+\varepsilon)w & \text{if } w \le -\rho, \\ g(m) + (\beta+2\varepsilon)w & \text{if } w \in (-\rho,\rho), \\ g(m) - C + (\beta+\kappa+\varepsilon)w & \text{if } w \ge \rho, \end{cases} \qquad (m,w) \in [0,1]\times\mathbb{R}.$$

The main challenge here is that no matter how we define $F_{\to}$ at $w=-\rho$ and $w=\rho$, there always is a discontinuity at these points. On the other hand, we observe that $F_{\leftarrow}$ is in fact continuous since $C = (\kappa-\varepsilon)\rho$ implies

$$C + (\beta+\kappa+\varepsilon)w\Big|_{w=-\rho} = (\kappa-\varepsilon)\rho - (\beta+\kappa+\varepsilon)\rho = -(\beta+2\varepsilon)\rho = (\beta+2\varepsilon)w\Big|_{w=-\rho}$$

and, similarly,

$$-C + (\beta+\kappa+\varepsilon)w\Big|_{w=\rho} = -(\kappa-\varepsilon)\rho + (\beta+\kappa+\varepsilon)\rho = (\beta+2\varepsilon)\rho = (\beta+2\varepsilon)w\Big|_{w=\rho}.$$

To handle the discontinuity of $F_{\to}$, we introduce the regularized forward operator

$$F^{\lambda}_{\to}(m,w) := \varepsilon + (\kappa-\varepsilon)\phi^{\lambda}_1(w) - \big(2\varepsilon + (\kappa-\varepsilon)\phi^{\lambda}_2(w)\big)m, \qquad (m,w) \in [0,1]\times\mathbb{R},$$

where for $\lambda \in (0,\rho]$ we let $\phi^{\lambda}_1, \phi^{\lambda}_2 : \mathbb{R} \to [0,1]$ be the Lipschitz continuous functions

$$\phi^{\lambda}_1(w) = \begin{cases} 0 & \text{if } w \le \rho-\lambda \\ \frac{1}{\lambda}(w-\rho+\lambda) & \text{if } w \in (\rho-\lambda,\rho) \\ 1 & \text{if } w \ge \rho \end{cases}$$

and

$$\phi^{\lambda}_2(w) = \begin{cases} 1 & \text{if } w \le -\rho \\ -\frac{1}{\lambda}(w+\rho-\lambda) & \text{if } w \in (-\rho,-\rho+\lambda) \\ 0 & \text{if } w \in [-\rho+\lambda,\rho-\lambda] \\ \frac{1}{\lambda}(w-\rho+\lambda) & \text{if } w \in (\rho-\lambda,\rho) \\ 1 & \text{if } w \ge \rho, \end{cases}$$

respectively. Note that

$$\lim_{\lambda\downarrow 0}\phi^{\lambda}_1(w) = \begin{cases} 0 & \text{if } w < \rho \\ 1 & \text{if } w \ge \rho \end{cases} \qquad \text{and} \qquad \lim_{\lambda\downarrow 0}\phi^{\lambda}_2(w) = \begin{cases} 1 & \text{if } w \le -\rho \\ 0 & \text{if } w \in (-\rho,\rho) \\ 1 & \text{if } w \ge \rho, \end{cases}$$

from which we conclude that

$$\lim_{\lambda\downarrow 0} F^{\lambda}_{\to}(m,w) = F_{\to}(m,w) \qquad \text{for all } m \in [0,1] \text{ and } w \in \mathbb{R}\setminus\{-\rho,\rho\}.$$

We proceed to show that the regularized equilibrium system

$$\dot{M}^\lambda(t) = F^\lambda_\rightarrow\big(M^\lambda(t), W^\lambda(t)\big), \qquad t \in [0,T], \qquad M^\lambda(0) = M_0, \tag{A.1}$$

$$\dot{W}^\lambda(t) = F_\leftarrow\big(M^\lambda(t), W^\lambda(t)\big), \qquad t \in [0,T], \qquad W^\lambda(T) = G\big(M^\lambda(T)\big), \tag{A.2}$$

admits a solution (M^λ, W^λ) which converges to a solution (M, W) of the original, unregularized equilibrium system as $\lambda \downarrow 0$.

Proposition A.1. *For each $\lambda \in (0, \rho]$, there exist continuously differentiable functions*

$$M^\lambda : [0,T] \to [0,1] \qquad \textit{and} \qquad W^\lambda : [0,T] \to \mathbb{R}$$

which solve the regularized equilibrium system (A.1) – (A.2)*. Moreover, the family of functions $\{(M^\lambda, W^\lambda)\}_{\lambda \in (0,\rho]}$ is uniformly bounded and uniformly Lipschitz continuous on $[0,T]$.* $\diamond$

Proof. The existence of M^λ and W^λ follows from Schauder's fixed point theorem; see for example Appendix A in [BHS21] for the line of argument in a slightly more complicated setting. The uniform boundedness of M^λ is obvious since M^λ is $[0,1]$-valued, whereas the uniform boundedness of W^λ follows from the bounds

$$F_\leftarrow\big(m, w\big) \leq F_{down}(w) := g(0) + C + (\beta + \kappa + \varepsilon)w, \qquad (m, w) \in [0,1] \times \mathbb{R},$$

and, similarly,

$$F_\leftarrow\big(m, w\big) \geq F_{up}(w) := g(1) - C + (\beta + 2\varepsilon)w, \qquad (m, w) \in [0,1] \times \mathbb{R},$$

where we have used that g is decreasing. To make the argument precise, let us introduce two functions $W_{down} : [0,T] \to \mathbb{R}$ and $W_{up} : [0,T] \to \mathbb{R}$ given as the unique solutions of the linear differential equations

$$\dot{W}_{down}(t) = F_{down}\big(W_{down}(t)\big), \qquad t \in [0,T], \qquad W_{down}(T) = \min_{m \in [0,1]} G(m),$$

$$\dot{W}_{up}(t) = F_{up}\big(W_{up}(t)\big), \qquad t \in [0,T], \qquad W_{up}(T) = \max_{m \in [0,1]} G(m).$$

Then we have $W_{down}(T) \leq W^\lambda(T) \leq W_{up}(T)$ and

$$\dot{W}_{up}(t) - F_{up}\big(W_{up}(t)\big) \leq W^\lambda(t) - F_{up}\big(W^\lambda(t)\big), \qquad t \in [0,T]$$

$$\dot{W}^\lambda(t) - F_{down}\big(W^\lambda(t)\big) \leq \dot{W}_{down}(t) - F_{down}\big(W_{down}(t)\big) \qquad t \in [0,T].$$

We now argue that this implies

$$W_{down}(t) \leq W^\lambda(t) \leq W_{up}(t), \qquad t \in [0,T] :$$

As usual we only prove the first statement, i.e., $W_{down}(t) \leq W^\lambda(t)$ for all $t \in [0,T]$. Assume not, then there are $\hat{t} \in [0,T]$ and $\epsilon > 0$ such that $W_{down}(\hat{t}) = W^\lambda(\hat{t})$ and $W_{down}(t) > W^\lambda(t)$ for all $t \in (\hat{t} - \epsilon, \hat{t})$. In this case $\Delta(t) := W_{down}(t) - W^\lambda(t) \geq 0$ for all $t \in (\hat{t} - \epsilon, \hat{t}]$ and

$$\dot{\Delta}(t) = \dot{W}_{down}(t) - \dot{W}^\lambda(t) \geq F_{down}(W_{down}(t)) - F_{down}(W^\lambda(t))$$

$$= (\beta + \kappa + \epsilon)\Delta(t), \qquad\qquad t \in (\hat{t} - \epsilon, \hat{t}).$$

Hence, $\Delta'(t) := \Delta(t)e^{-(\beta+\kappa+\epsilon)t}$ satisfies $\dot{\Delta}'(t) \geq 0$ for all $t \in (\hat{t} - \epsilon, \hat{t})$. This implies that $\Delta'(t) \leq \Delta'(\hat{t}) = 0$ for all $t \in (\hat{t} - \epsilon, \hat{t})$. Therefore, we obtain $\Delta(t) \leq 0$ for all $t \in (\hat{t} - \epsilon, \hat{t})$, which is the desired contradiction.

Since W_{down} and W_{up} do not depend on λ, the uniform bound on W^λ obtains. Regarding the uniform Lipschitz continuity of M^λ and W^λ, it suffices to observe that both functions are continuously differentiable and

$$F_{\to}^{\lambda} \text{ and } F_{\leftarrow} \text{ are bounded on } [0,1] \times \left[\min_{t\in[0,T]} W_{down}(t), \max_{t\in[0,T]} W_{up}(t)\right]$$

with a bound not depending on λ. □

We can now employ an argument based on the Arzelà-Ascoli theorem and Komlós' lemma to construct a sequence of solutions of the regularized system which converges to a pair of functions (M, W) such that M satisfies the forward equation in the sense of Fillipov, that is

$$\dot{M}(t) \in \begin{cases} \{F_{\to}(M(t), W(t))\} & \text{if } W(t) \neq -\rho, \rho, \\ [\varepsilon - (\kappa+\varepsilon)M(t), \varepsilon - 2\varepsilon M(t)] & \text{if } W(t) = -\rho, \\ [\varepsilon - 2\varepsilon M(t), \kappa - (\kappa+\varepsilon)M(t)] & \text{if } W(t) = \rho, \end{cases} \qquad \text{for a.e. } t \in [0,T], \tag{A.3}$$

whereas W is the desired solution of the backward equation, that is,

$$\dot{W}(t) = F_{\leftarrow}(M(t), W(t)) \qquad \text{for all } t \in [0,T]. \tag{A.4}$$

Proposition A.2. *There exists a pair of functions (M, W) such that $M : [0,T] \to [0,1]$ is absolutely continuous and satisfies* (A.3) *with $M(0) = M_0$ and $W : [0,T] \to \mathbb{R}$ is continuously differentiable and satisfies* (A.4) *with $W(T) = G(M(T))$.* ⋄

Proof. Step 1: Construction of a candidate solution. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, \rho]$ be a sequence converging to zero. For each $n \in \mathbb{N}$, let (M^n, W^n) be the solution of the regularized equilibrium system associated with λ_n given by Proposition A.1. With a slight abuse of notation, we write

$$\phi_1^n := \phi_1^{\lambda_n} \qquad \text{and} \qquad \phi_2^n := \phi_2^{\lambda_n} \qquad \text{and} \qquad F_{\to}^n := F_{\to}^{\lambda_n}$$

and set

$$\phi_1^\infty(w) := \begin{cases} 0 & \text{if } w < \rho \\ 1 & \text{if } w \geq \rho \end{cases} \qquad \text{and} \qquad \phi_2^\infty(w) := \begin{cases} 1 & \text{if } w \leq -\rho \\ 0 & \text{if } w \in (-\rho, \rho) \\ 1 & \text{if } w \geq \rho, \end{cases}$$

so that $(\phi_1^n, \phi_2^n) \to (\phi_1^\infty, \phi_2^\infty)$ pointwise as $n \to \infty$. Since each pair (M^n, W^n), $n \in \mathbb{N}$, shares the same Lipschitz constant and the same bound, we see that $\{(M^n, W^n)\}_{n\in\mathbb{N}}$ is equicontinuous and pointwise bounded. Therefore, it follows from the Arzelà-Ascoli theorem that a subsequence, which for convenience is again denoted by $\{(M^n, W^n)\}_{n\in\mathbb{N}}$, converges uniformly on $[0,T]$

to a pair of continuous functions $(M, \tilde{W}) : [0, T] \to [0, 1] \times \mathbb{R}$. Next, note that the sequences $\{\phi_1^n(W^n)\}_{n\in\mathbb{N}}$ and $\{\phi_2^n(W^n)\}_{n\in\mathbb{N}}$ are bounded, hence in particular bounded in $L^2([0, T])$, and therefore Komlós' lemma [DS06, Theorem 15.1.2] guarantees the existence of a sequence $\{(p_1^n, p_2^n)\}_{n\in\mathbb{N}}$ of functions such that

$$(p_1^n, p_2^n) \in \operatorname{conv}\Big\{ \big(\phi_1^n(W^n), \phi_2^n(W^n)\big), \big(\phi_1^{n+1}(W^{n+1}), \phi_2^{n+1}(W^{n+1})\big), \dots \Big\} \quad \text{for all } n \in \mathbb{N}$$

converging in $L^2([0, T])$ to a pair of functions $(p_1, p_2) : [0, T] \to [0, 1]^2$. But then there is a subsequence of $\{(p_1^n, p_2^n)\}_{n\in\mathbb{N}}$ that converges for a.e. $t \in [0, T]$ to (p_1, p_2). Just as before, we still denote this subsequence by $\{(p_1^n, p_2^n)\}_{n\in\mathbb{N}}$. We highlight that, for each $n \in \mathbb{N}$, there are $L(n) \in \mathbb{N}$ convex weights $\omega_1, \dots, \omega_{L(n)} \in (0, 1]$ and natural numbers $n_1 < \dots < n_{L(n)}$ with $n_1 \geq n$ such that

$$p_1^n = \sum_{\ell=1}^{L(n)} \omega_\ell \phi_1^{n_\ell}(W^{n_\ell}) \qquad \text{and} \qquad p_2^n = \sum_{\ell=1}^{L(n)} \omega_\ell \phi_2^{n_\ell}(W^{n_\ell}).$$

Setting

$$\tilde{M}^n := \sum_{\ell=1}^{L(n)} \omega_\ell M^{n_\ell} \qquad \text{and} \qquad \tilde{W}^n := \sum_{\ell=1}^{L(n)} \omega_\ell W^{n_\ell},$$

it follows from $(M^n, W^n) \to (M, \tilde{W})$ that also $(\tilde{M}^n, \tilde{W}^n) \to (M, \tilde{W})$ uniformly on $[0, T]$. With this, let us now define $W : [0, T] \to \mathbb{R}$ to be the unique continuously differentiable solution of

$$\dot{W}(t) = F_{\leftarrow}\big(M(t), W(t)\big), \qquad t \in [0, T], \qquad W(T) = G\big(M(T)\big).$$

Step 2: We argue that $\tilde{W} = W$. For each $n \in \mathbb{N}$ and each $t \in [0, T]$, we note that

$$\begin{aligned}
&\big|W^n(t) - W(t)\big| \\
&\leq \big|G\big(M^n(T)\big) - G\big(M(T)\big)\big| + \int_t^T \big|F_{\leftarrow}\big(M^n(s), W^n(s)\big) - F_{\leftarrow}\big(M(s), W(s)\big)\big| \, \mathrm{d}s \\
&\leq \big|G\big(M^n(T)\big) - G\big(M(T)\big)\big| + LT \sup_{t\in[0,T]} \big|M^n(t) - M(t)\big| + L \int_t^T \big|W^n(s) - W(s)\big| \, \mathrm{d}s,
\end{aligned}$$

where we have used that $F_{\leftarrow}$ is Lipschitz continuous with Lipschitz constant denoted by $L > 0$. Gronwall's inequality implies

$$\big|W^n(t) - W(t)\big| \leq \Big(\big|G\big(M^n(T)\big) - G\big(M(T)\big)\big| + LT \sup_{t\in[0,T]} \big|M^n(t) - M(t)\big| \Big) e^{LT},$$

from which we conclude $W^n \to W$ pointwise as $n \to \infty$. But $W^n \to \tilde{W}$ as $n \to \infty$ by definition of $\tilde{W}$, so we must have $W = \tilde{W}$.

Step 3: We are left with showing that M is absolutely continuous and satisfies (A.3). For this, let us observe that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{M}^n(t) = \lim_{n\to\infty} \sum_{\ell=1}^{L(n)} \omega_\ell \dot{M}^{n_\ell}(t) = \lim_{n\to\infty} \sum_{\ell=1}^{L(n)} \omega_\ell F_{\rightarrow}^{n_\ell}\big(M^{n_\ell}(t), W^{n_\ell}(t)\big) \\
&= \lim_{n\to\infty} \sum_{\ell=1}^{L(n)} \omega_\ell \Big[\varepsilon + (\kappa-\varepsilon)\phi_1^{n_\ell}\big(W^{n_\ell}(t)\big) - 2\varepsilon M^{n_\ell}(t) - (\kappa-\varepsilon)\phi_2^{n_\ell}\big(W^{n_\ell}(t)\big)M^{n_\ell}(t)\Big] \\
&= \varepsilon + \lim_{n\to\infty}\Big[(\kappa-\varepsilon)p_1^{n_\ell}(t) - 2\varepsilon\tilde{M}^{n_\ell}(t) - (\kappa-\varepsilon)p_2^{n_\ell}(t)M(t) \\
&\qquad\qquad - (\kappa-\varepsilon)\sum_{\ell=1}^{L(n)} \omega_\ell \phi_2^{n_\ell}\big(W^{n_\ell}(t)\big)\big(M^{n_\ell}(t) - M(t)\big)\Big] \\
&= \varepsilon + (\kappa-\varepsilon)p_1(t) - \big(2\varepsilon + (\kappa-\varepsilon)p_2(t)\big)M(t)
\end{aligned}$$

for almost every $t \in [0,T]$. From this, it follows by dominated convergence that

$$\begin{aligned}
M(t) &= \lim_{n\to\infty} \tilde{M}^n(t) = M(0) + \lim_{n\to\infty} \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\tilde{M}^n(s)\,\mathrm{d}s \\
&= M(0) + \int_0^t \big[\varepsilon + (\kappa-\varepsilon)p_1(s) - \big(2\varepsilon + (\kappa-\varepsilon)p_2(s)\big)M(s)\big]\,\mathrm{d}s
\end{aligned}$$

for all $t \in [0,T]$, so M is indeed absolutely continuous and its a.e. derivative satisfies

$$\dot{M}(t) = \varepsilon + (\kappa-\varepsilon)p_1(t) - \big(2\varepsilon + (\kappa-\varepsilon)p_2(t)\big)M(t) \qquad \text{for a.e. } t \in [0,T].$$

Observe that

$$\tilde{W}(t) \neq \rho \text{ implies the existence of } N \in \mathbb{N} \text{ such that } p_1(t) = \phi_1^n\big(\tilde{W}^n(t)\big) \text{ for all } n \geq N$$

and, similarly,

$$\tilde{W}(t) \neq -\rho, \rho \text{ implies the existence of } N \in \mathbb{N} \text{ such that } p_2(t) = \phi_2^n\big(\tilde{W}^n(t)\big) \text{ for all } n \geq N.$$

Hence, we can assume without loss of generality

$$(p_1, p_2) = \big(\phi_1^\infty(\tilde{W}), \phi_2^\infty(\tilde{W})\big) = \big(\phi_1^\infty(W), \phi_2^\infty(W)\big) \qquad \text{whenever } W \neq -\rho, \rho,$$

from which we conclude that

$$\dot{M}(t) = F_{\rightarrow}\big(M(t), W(t)\big) \qquad \text{for a.e. } t \in [0,T] \text{ with } W(t) \neq -\rho, \rho.$$

Since $\phi_1^\lambda = \phi_2^\lambda$ on $[-\rho+\lambda, \infty)$ for all $\lambda \in (0,\rho)$, we have that $p_1^n = p_2^n$ on $\{W(t) = \rho\}$ for eventually all $n \in \mathbb{N}$. Hence, we can assume without loss of generality that $p_1 = p_2 \in [0,1]$ on $\{W(t) = \rho\}$, implying that

$$\dot{M}(t) \in \big[\varepsilon - 2\varepsilon M(t), \kappa - (\kappa+\varepsilon)M(t)\big] \qquad \text{for a.e. } t \in [0,T] \text{ with } W(t) = \rho.$$

Similarly, we note that $\phi_1^\lambda = 0$ on $(-\infty, \rho - \lambda)$ for all $\lambda \in (0, \rho)$, which in turn implies $p_1^n = 0$ on $\{W(t) = -\rho\}$ for eventually all $n \in \mathbb{N}$. Hence, we can assume without loss of generality that $p_1 = 0$ and $p_2 \in [0, 1]$ on $\{W(t) = -\rho\}$, so that

$$\dot{M}(t) \in \big[\varepsilon - (\kappa + \varepsilon)M(t), \varepsilon - 2\varepsilon M(t)\big] \qquad \text{for a.e. } t \in [0, T] \text{ with } W(t) = -\rho.$$

Thus M satisfies (A.3). □

Since W is continuous, it follows that $\{W \notin \{-\rho, \rho\}\}$ is open relative to $[0, T]$ and hence decomposes into a family $\{I_n\}_{n \in \hat{\mathcal{N}}}$, $\hat{\mathcal{N}} \subseteq \mathbb{N}$, of disjoint intervals which are open relative to $[0, T]$. As such, by passing to the complement, we can write

$$\big\{W \in \{-\rho, \rho\}\big\} = \bigcup_{n \in \hat{\mathcal{N}} \cup \{0\}} A_n,$$

where $\{A_n\}_{n \in \hat{\mathcal{N}} \cup \{0\}}$ is a sequence of closed intervals. On each I_n, $n \in \hat{\mathcal{N}}$, the right-hand side of (A.3) is a singleton, so we have

$$\dot{M}(t) = F_{\rightarrow}\big(M(t), W(t)\big) \qquad \text{for a.e. } t \in I_n,$$

and since $W \neq -\rho, \rho$ on I_n, the right-hand side never switches cases on I_n. Thus M satisfies a linear differential equation on I_n and can be chosen to be continuously differentiable on I_n. On the other hand, we can in general not rule out that all A_n, $n \in \hat{\mathcal{N}} \cup \{0\}$, have an empty interior, which means that, in general, we cannot argue that $\{W \in \{-\rho, \rho\}\}$ has Lebesgue measure zero. In fact, this question depends crucially on the parameters k_1 and k_2 defined in Equation (2.6).

Lemma A.3. *For $n \in \hat{\mathcal{N}} \cup \{0\}$, suppose that A_n has a non-empty interior denoted by (a, b) for some $a, b \in [0, T]$ with $a < b$. Then M is constant and continuously differentiable on (a, b) with*

$$\begin{aligned} \big(M, \dot{M}\big) &= (k_1, 0) \quad \text{on } (a, b) && \text{if} && W = -\rho \text{ on } (a, b), \\ \big(M, \dot{M}\big) &= (k_2, 0) \quad \text{on } (a, b) && \text{if} && W = \rho \text{ on } (a, b). \end{aligned}$$

Moreover, it holds that

$$k_1 \notin \left[\tfrac{\varepsilon}{\kappa + \varepsilon}, \tfrac{1}{2}\right] \text{ implies that any } A_n \text{ with } W = -\rho \text{ on } A_n \text{ is a singleton},$$

and, similarly,

$$k_2 \notin \left[\tfrac{1}{2}, \tfrac{\kappa}{\kappa + \varepsilon}\right] \text{ implies that any } A_n \text{ with } W = \rho \text{ on } A_n \text{ is a singleton.}$$ ◇

Proof. By definition of A_n and continuity of W must have $W = -\rho$ or $W = \rho$ on all of (a, b), so W is constant and satisfies $\dot{W} = 0$ on (a, b). Let us consider the case $W = -\rho$, the other case follows by similar arguments. Since $W = -\rho$ on all of (a, b), we find that

$$0 = \dot{W}(t) = g\big(M(t)\big) + C + (\beta + \kappa + \varepsilon)W(t) = g\big(M(t)\big) - (\beta + 2\varepsilon)\rho, \qquad t \in (a, b),$$

which is to say that

$$M(t) = g^{-1}\big((\beta + 2\varepsilon)\rho\big) = k_1, \qquad t \in (a,b).$$

In particular, M is constant and hence continuously differentiable with $\dot{M} = 0$ on (a,b). But since $W = -\rho$ and $M = k_1$, this is only possible if

$$0 = \dot{M}(t) \in \big[\varepsilon - (\kappa+\varepsilon)M(t), \varepsilon - 2\varepsilon M(t)\big] = \big[\varepsilon - (\kappa+\varepsilon)k_1, \varepsilon - 2\varepsilon k_1\big], \quad \text{for a.e. } t \in (a,b).$$

The last statement is equivalent to $k_1 \in [\varepsilon/(\kappa+\varepsilon), 1/2]$, which therefore concludes the proof. ☐

Combining what we have established thus far, we arrive at the following existence result for the equilibrium system.

Theorem A.4. *There exist functions*

$$M : [0,T] \to [0,1] \qquad \textit{and} \qquad W : [0,T] \to \mathbb{R}$$

where M is absolutely continuous, W is continuously differentiable and (M,W) solves the system (2.2) – (2.3). *Moreover, for any solution (M,W) of this system there exists $\mathcal{N} \subseteq \mathbb{N}$ and disjoint open intervals $\{I_n\}_{n\in\mathcal{N}}$ with $\bigcup_{n\in\mathcal{N}} I_n$ dense in $[0,T]$ such that, for each $n \in \mathcal{N}$, M is continuously differentiable on I_n and one of the following five cases holds true:*

$$\begin{array}{rllll}
(\mathrm{Lo}) & W < -\rho & \textit{and} & \dot{M} = \varepsilon - (\kappa+\varepsilon)M & \textit{on } I_n, \\
(\mathrm{Med}) & W \in (-\rho,\rho) & \textit{and} & \dot{M} = \varepsilon - 2\varepsilon M & \textit{on } I_n, \\
(\mathrm{Hi}) & W > \rho & \textit{and} & \dot{M} = \kappa - (\kappa+\varepsilon)M & \textit{on } I_n, \\
(\mathrm{Eq}^-) & W = -\rho & \textit{and} & M = k_1 & \textit{on } I_n, \\
(\mathrm{Eq}^+) & W = \rho & \textit{and} & M = k_2 & \textit{on } I_n.
\end{array}$$

In particular, there are at most countably many points at which M is not continuously differentiable. Finally,

$$(\mathrm{Eq}^-) \textit{ is only possible if } k_1 \in \Big[\tfrac{\varepsilon}{\kappa+\varepsilon}, \tfrac{1}{2}\Big] \quad \textit{and} \quad (\mathrm{Eq}^+) \textit{ is only possible if } k_2 \in \Big[\tfrac{1}{2}, \tfrac{\kappa}{\kappa+\varepsilon}\Big]. \quad \diamond$$

Let us highlight that in the cases (Eq^-) and (Eq^+), we know M and W explicitly. In the other cases, both M and W satisfy linear differential equations which we can solve in closed form. For this, let us fix $n \in \mathcal{N}$ and write $I_n = (a,b)$ for suitable $a, b \in [0,T]$ with $a < b$.
Case (Lo). If I_n is of type (Lo), the pair (M,W) solves

$$\begin{aligned}
\dot{M}(t) &= \varepsilon - (\kappa+\varepsilon)M(t), && t \in (a,b), \\
\dot{W}(t) &= g\big(M(t)\big) + C + (\beta+\kappa+\varepsilon)W(t), && t \in (a,b).
\end{aligned}$$

From this, we find that $(M,W) = (M^{(\mathrm{Lo})}, W^{(\mathrm{Lo})})$ on $[a,b]$, where, for all $t \in [a,b]$,

$$M^{(\mathrm{Lo})}(t) = \Big(M(a) - \tfrac{\varepsilon}{\kappa+\varepsilon}\Big)e^{-(\kappa+\varepsilon)(t-a)} + \tfrac{\varepsilon}{\kappa+\varepsilon} \tag{A.5}$$

$$= \Big(M(b) - \tfrac{\varepsilon}{\kappa+\varepsilon}\Big) e^{(\kappa+\varepsilon)(b-t)} + \tfrac{\varepsilon}{\kappa+\varepsilon}, \tag{A.6}$$

$$\begin{aligned} W^{(\mathrm{Lo})}(t) &= W(a) e^{(\beta+\kappa+\varepsilon)(t-a)} + \int_a^t e^{(\beta+\kappa+\varepsilon)(t-s)} \big[g\big(M^{(\mathrm{Lo})}(s)\big) + C\big] \,\mathrm{d}s \\ &= W(b) e^{-(\beta+\kappa+\varepsilon)(b-t)} - \int_t^b e^{(\beta+\kappa+\varepsilon)(t-s)} \big[g\big(M^{(\mathrm{Lo})}(s)\big) + C\big] \,\mathrm{d}s. \end{aligned} \tag{A.7}$$

Case (Med). If I_n is of type (Med), the pair (M, W) solves

$$\begin{aligned} \dot{M}(t) &= \varepsilon - 2\varepsilon M(t), && t \in (a,b), \\ \dot{W}(t) &= g\big(M(t)\big) + (\beta + 2\varepsilon) W(t), && t \in (a,b), \end{aligned}$$

From this, we find that $(M, W) = (M^{(\mathrm{Med})}, W^{(\mathrm{Med})})$ on $[a,b]$, where, for all $t \in [a,b]$,

$$M^{(\mathrm{Med})}(t) = \Big(M(a) - \tfrac{1}{2}\Big) e^{-2\varepsilon(t-a)} + \tfrac{1}{2} \tag{A.8}$$

$$= \Big(M(b) - \tfrac{1}{2}\Big) e^{2\varepsilon(b-t)} + \tfrac{1}{2}, \tag{A.9}$$

$$\begin{aligned} W^{(\mathrm{Med})}(t) &= W(a) e^{(\beta+2\varepsilon)(t-a)} + \int_a^t e^{(\beta+2\varepsilon)(t-s)} g\big(M^{(\mathrm{Med})}(s)\big) \,\mathrm{d}s \\ &= W(b) e^{-(\beta+2\varepsilon)(b-t)} - \int_t^b e^{(\beta+2\varepsilon)(t-s)} g\big(M^{(\mathrm{Med})}(s)\big) \,\mathrm{d}s. \end{aligned} \tag{A.10}$$

Case (Hi). If I_n is of type (Hi), the pair (M, W) solves

$$\begin{aligned} \dot{M}(t) &= \kappa - (\kappa+\varepsilon) M(t), && t \in (a,b), \\ \dot{W}(t) &= g\big(M(t)\big) - C + (\beta+\kappa+\varepsilon) W(t), && t \in (a,b). \end{aligned}$$

From this, we find that $(M, W) = (M^{(\mathrm{Hi})}, W^{(\mathrm{Hi})})$ on $[a,b]$, where, for all $t \in [a,b]$,

$$M^{(\mathrm{Hi})}(t) = \Big(M(a) - \tfrac{\kappa}{\kappa+\varepsilon}\Big) e^{-(\kappa+\varepsilon)(t-a)} + \tfrac{\kappa}{\kappa+\varepsilon} \tag{A.11}$$

$$= \Big(M(b) - \tfrac{\kappa}{\kappa+\varepsilon}\Big) e^{(\kappa+\varepsilon)(b-t)} + \tfrac{\kappa}{\kappa+\varepsilon}, \tag{A.12}$$

$$\begin{aligned} W^{(\mathrm{Hi})}(t) &= W(a) e^{(\beta+\kappa+\varepsilon)(t-a)} + \int_a^t e^{(\beta+\kappa+\varepsilon)(t-s)} \big[g\big(M^{(\mathrm{Hi})}(s)\big) - C\big] \,\mathrm{d}s \\ &= W(b) e^{-(\beta+\kappa+\varepsilon)(b-t)} - \int_t^b e^{(\beta+\kappa+\varepsilon)(t-s)} \big[g\big(M^{(\mathrm{Hi})}(s)\big) - C\big] \,\mathrm{d}s. \end{aligned}$$

B. Construction of Dynamic Equilibria

This section is devoted to the proof of Theorem 2.1. Let us first recall the representative agent's optimization problem given by

$$\sup_{\nu \in \mathcal{A}_T} \mathbb{E}^{\nu}\Big[\int_0^T e^{-\beta t} \psi_{X_t}(M(t), \nu_t) \,\mathrm{d}t + e^{-\beta T} \Psi_{X_T}\big(M(T)\big)\Big].$$

Following [BHS21], we denote by

$$\mathcal{H} : \mathbb{S} \times [0,T] \times [0,1] \times \mathbb{R}^2 \to \mathbb{R},$$
$$(i,t,m,v) \mapsto \mathcal{H}(i,t,m,v) := \max_{u\in\mathbb{U}} \Big\{ e^{-\beta t}\psi_i(m,u) + Q_{i\cdot}(u)v \Big\},$$

the Hamiltonian associated with this optimization problem, where we write $Q_{i\cdot}(u)$ as shorthand notation for the i-th row of the transition rate matrix $Q(u)$. Next, we compute a maximizer of the Hamiltonian, i.e. a function

$$h^* : [0,T] \times [0,1] \times \mathbb{R}^2 \to \mathbb{U}^2, \qquad (t,m,v) \mapsto h^*(t,m,v) = \begin{pmatrix} h_1^*(t,m,v) \\ h_2^*(t,m,v) \end{pmatrix}$$

such that

$$\mathcal{H}(i,t,m,v) = e^{-\beta t}\psi_i\big(m, h_i^*(t,m,v)\big) + Q_{i\cdot}\big(h_i^*(t,m,v)\big)v$$

for all $(i,t,m,v) \in \mathbb{S} \times [0,T] \times [0,1] \times \mathbb{R}^2$. It is easily checked that such a function h^* is given by

$$h^* : [0,T] \times \mathbb{R}^2 \to \mathbb{U}^2, \quad (t,v) \mapsto h^*(t,v) := \begin{cases} (switch, stay) & \text{if } (v_1 - v_2)e^{\beta t} \le -\rho \\ (stay, stay) & \text{if } (v_1 - v_2)e^{\beta t} \in (-\rho, \rho) \\ (stay, switch) & \text{if } (v_1 - v_2)e^{\beta t} \ge \rho, \end{cases}$$

where we observe that in our particular case h^* does not depend on m. Given the maximizer h^*, we may now introduce the reduced-form transition rate matrix

$$\hat{Q} : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^{2\times 2},$$
$$(t,v) \mapsto \hat{Q}(t,v) := \begin{pmatrix} Q_{11}\big(h_1^*(t,v)\big) & Q_{12}\big(h_1^*(t,v)\big) \\ Q_{21}\big(h_2^*(t,v)\big) & Q_{22}\big(h_2^*(t,v)\big) \end{pmatrix},$$

and the reduced-form running reward

$$\hat{\psi} : [0,T] \times [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2, \qquad (t,m,v) \mapsto \hat{\psi}(t,m,v) := \begin{pmatrix} \psi_1\big(m, h_1^*(t,v)\big) \\ \psi_2\big(m, h_2^*(t,v)\big) \end{pmatrix} e^{-\beta t}.$$

With this, the equilibrium system derived in [BHS21] reads

$$\begin{aligned} \dot{M}(t) &= M(t)\hat{Q}_{11}\big(t, v(t)\big) + \big(1 - M(t)\big)\hat{Q}_{21}\big(t, v(t)\big), & M(0) &= M_0, \\ \dot{v}(t) &= -\hat{\psi}\big(t, M(t), v(t)\big) - \hat{Q}\big(t, v(t)\big)v(t), & v(T) &= e^{-\beta T}\Psi\big(M(T)\big), \end{aligned}$$

for any initial fraction of customers $M_0 \in [0,1]$. By Theorems 6 and 9 in [BHS21], any solution to this system of equations gives rise to an equilibrium (ν, M) by choosing $\nu \in \mathcal{A}_T$ as

$$\nu : [0,T] \times \mathbb{S} \to \mathbb{U}, \qquad (t,i) \mapsto \nu(t,i) := h_i^*\big(t, v(t)\big).$$

We observe that the optimizer h^* only depends on the scaled difference $(v_1 - v_2)e^{\beta t}$. Thus, it is natural to consider the transformation

$$W : [0,T] \to \mathbb{R}, \qquad t \mapsto W(t) := \big[v_1(t) - v_2(t)\big]e^{\beta t}.$$

With this, a simple calculation shows that (M, W) solves the reduced equilibrium system

$$\dot{M}(t) = \begin{cases} \varepsilon - (\kappa+\varepsilon)M(t) & \text{if } W(t) \le -\rho, \\ \varepsilon - 2\varepsilon M(t) & \text{if } W(t) \in (-\rho,\rho), \\ \kappa - (\kappa+\varepsilon)M(t) & \text{if } W(t) \ge \rho, \end{cases} \qquad \text{for a.e. } t \in [0,T],$$

$$\dot{W}(t) = \begin{cases} g\big(M(t)\big) + C + (\beta+\kappa+\varepsilon)W(t) & \text{if } W(t) \le -\rho, \\ g\big(M(t)\big) + (\beta+2\varepsilon)W(t) & \text{if } W(t) \in (-\rho,\rho), \\ g\big(M(t)\big) - C + (\beta+\kappa+\varepsilon)W(t) & \text{if } W(t) \ge \rho, \end{cases} \qquad \text{for all } t \in [0,T],$$

which is equivalent to (2.2) – (2.3) under the assumption that

$$\big\{t \in [0,T] : W(t) = -\rho \text{ or } W(t) = \rho\big\} \text{ has Lebesgue measure zero.}$$

Let us now suppose that we are in the situation of Theorem 2.1, that is, we start with a solution (M, W) of the reduced equilibrium system (2.2) – (2.3) and let

$$h : \mathbb{R} \to \mathbb{U}^2, \qquad w \mapsto h(w) := \begin{cases} (switch, stay) & \text{if } w \le -\rho, \\ (stay, stay) & \text{if } w \in (-\rho,\rho), \\ (stay, switch) & \text{if } w \ge \rho. \end{cases}$$

Now define

$$\nu(t,i) := h_i\big(W(t)\big), \qquad (t,i) \in [0,T] \times \mathbb{S}.$$

Using that

$$h^*(t,v) = h\big((v_1 - v_2)e^{\beta t}\big), \qquad (t,v) \in [0,T] \times \mathbb{R}^2,$$

it is then easily checked that (ν, M) is indeed a dynamic equilibrium as claimed.

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